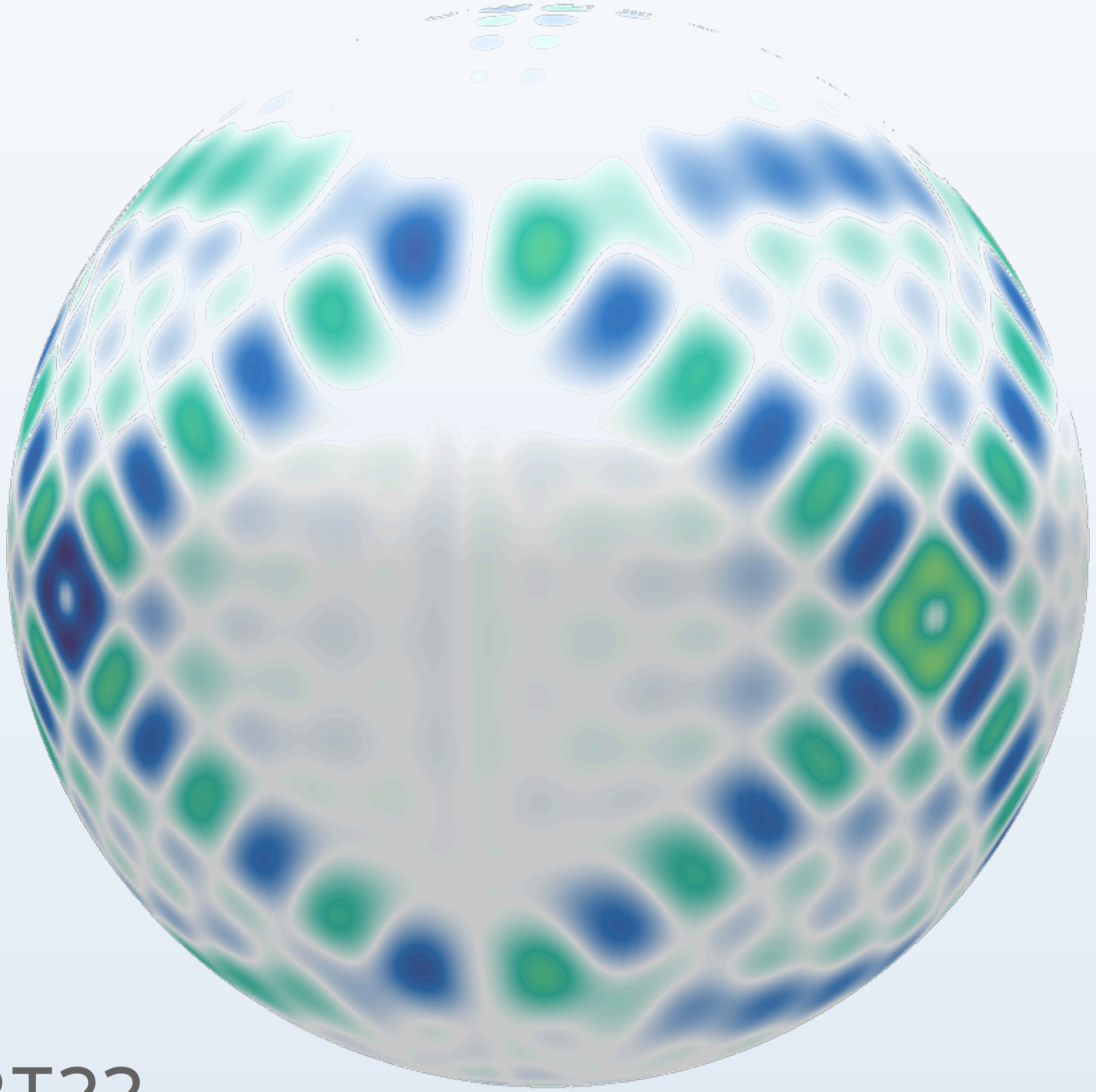
The two body density matrix of a Tomonaga-Luttinger liquid



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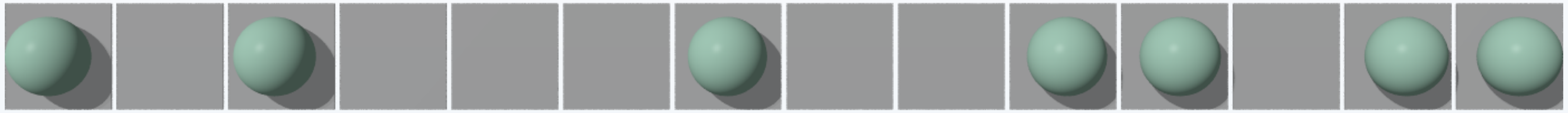
http://delmaestro.org/adrian • https://github.com/DelMaestroGroup/

M. Thamm, H. Radhakrishnan, H. Barghathi, B. Rosenow, AD, arXiv:2206.11301
H. Radhakrishnan, M. Thamm, H. Barghathi, B. Rosenow, AD, arXiv:2302.09093

Can we understand the interplay between interactions and (anti)symmetrization for strongly correlated itinerant particles?

Description of Itinerant Particles

N indistinguishable fermions on L sites $|\mathcal{H}| = \binom{L}{N}$



$$|\psi_\alpha\rangle = |10100010011011\rangle$$

$$= \frac{1}{\sqrt{7!}} \sum_{\mathcal{P}} (-1)^{\mathcal{P}} \mathcal{P} \, |2_1 3_2 7_3 10_4 11_5 13_6 14_7\rangle$$

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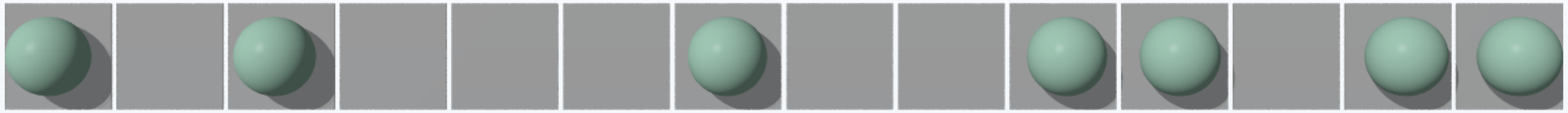

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$$= \frac{1}{\sqrt{7!}} \sum_{\mathcal{P}} (-1)^{\mathcal{P}} \mathcal{P} \, |2_1 3_2 7_3 10_4 11_5 13_6 14_7\rangle$$

general state: $$|\Psi\rangle = \sum_\alpha C_\alpha |\psi_\alpha\rangle$$

1st quantization: $$\Psi(i_1, \ldots, i_N) = \langle i_1, \ldots, i_N | \Psi\rangle$$

$i_\alpha \in \{1, L\}$

$$\Psi(i_1, \ldots, i_\mu, \ldots, i_\nu, \ldots i_N) = -\Psi(i_1, \ldots, i_\nu, \ldots, i_\mu, \ldots i_N)$$

density matrix: $\rho = |\Psi\rangle \langle\Psi|$ dimension: $|\mathcal{H}| \times |\mathcal{H}|$ $\langle\mathcal{O}\rangle = \mathrm{Tr}(\rho\mathcal{O})$

$\mathrm{Tr}\,\rho = 1$

Do we need the wavefunction?

REVIEWS OF MODERN PHYSICS VOLUME 32, NUMBER 2 APRIL, 1960

Present State of Molecular Structure Calculations*

C. A. Coulson
Mathematical Institute, Oxford, England

(6) One of the most vigorously pursued lines of research during the last few years has been the density matrix. It has frequently been pointed out that a conventional many-electron wave function tells us more than we need to know. All the necessary information required for the energy and for calculating the properties of molecules is embodied in the first- and second-order density matrices. These may, of course, be obtained from the wave function by a process of integration. But this is aesthetically unpleasing, and so attempts have been made, by Löwdin, McWeeny, and others, to work directly with these matrices. There is an instinctive feeling that matters such as electron-correlation should show up in the two-particle density matrix. But here we are confronted by a serious lack of success. We do know the conditions that must be satisfied by the many-electron wave function $\psi(1,2,\cdots n)$, but we still do not know the conditions

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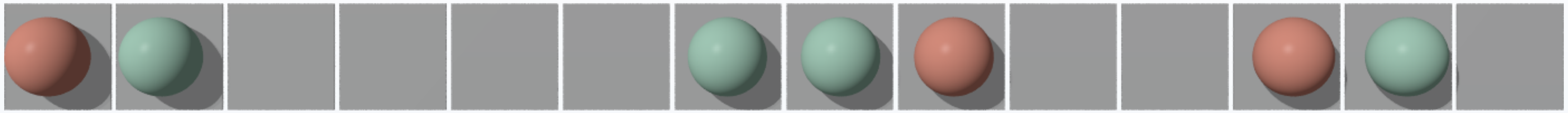
N-Representibility Problem

The *n*-particle density matrix must be constrained to represent a *N*-body density matrix (or wave function); otherwise, any minimized energy is unphysically below the ground-state energy for $N > 2$

A. J. Coleman, Rev. Mod. Phys. 35, 668 (1963)
D. A. Mazziotti, Phys. Rev. Lett. 108, 263002 (2011)
D. A. Mazziotti, Phys. Rev. Lett. 130, 153001 (2023)

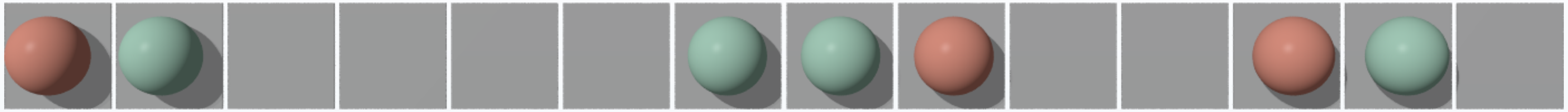
n-Particle Reduced Density Matrix (n-RDM)



$n = 3$
$N - n = 4$

Maximum information which is available about n particles, irrespective of the state of other N-n particles.

n-Particle Reduced Density Matrix (*n*-RDM)



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n-RDM: $\rho_n^{i_1,\ldots,i_n, j_1,\ldots,j_n} = \binom{N}{n}^{-1} \langle\Psi| c^\dagger_{i_1} \cdots c^\dagger_{i_n} c_{j_1} \cdots c_{j_n} |\Psi\rangle$ correlation function

$Tr\, \rho_n = 1$

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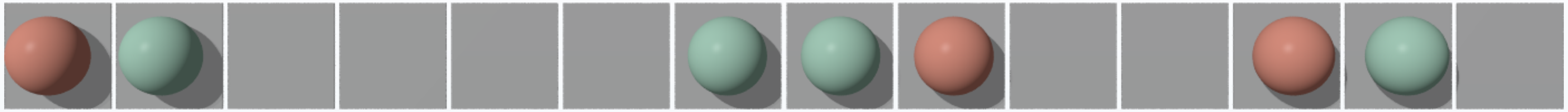

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integrate out

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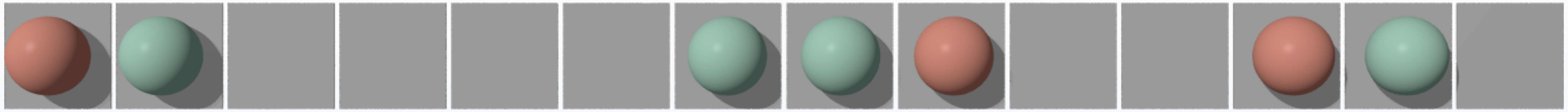

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integrate out

Any n-particle observable computable via n-particle reduced density matrix

$$\left\langle \mathcal{O}_{j_1,\ldots,j_n} \right\rangle = \sum_{j_1,\ldots,j_n} \left\{ \mathcal{O}_{j_1,\ldots j_n} \rho_n^{i_1,\ldots,i_n,j_1\ldots j_n} \right\}\Big|_{\substack{i_\alpha \to j_\alpha \\ \forall \alpha = 1,\ldots,n}}$$

1-Particle Reduced Density Matrix

1-RDM: $\rho_1^{i_1, j_1} = \frac{1}{N} \langle \Psi | c_{i_1}^\dagger c_{j_1} | \Psi \rangle$

$$= \sum_{i_2, \ldots, i_N} \Psi^*(i_1, \overbrace{i_2, \ldots, i_N}^{\text{integrate out}}) \Psi(j_1, \overbrace{i_2, \ldots, i_N})$$

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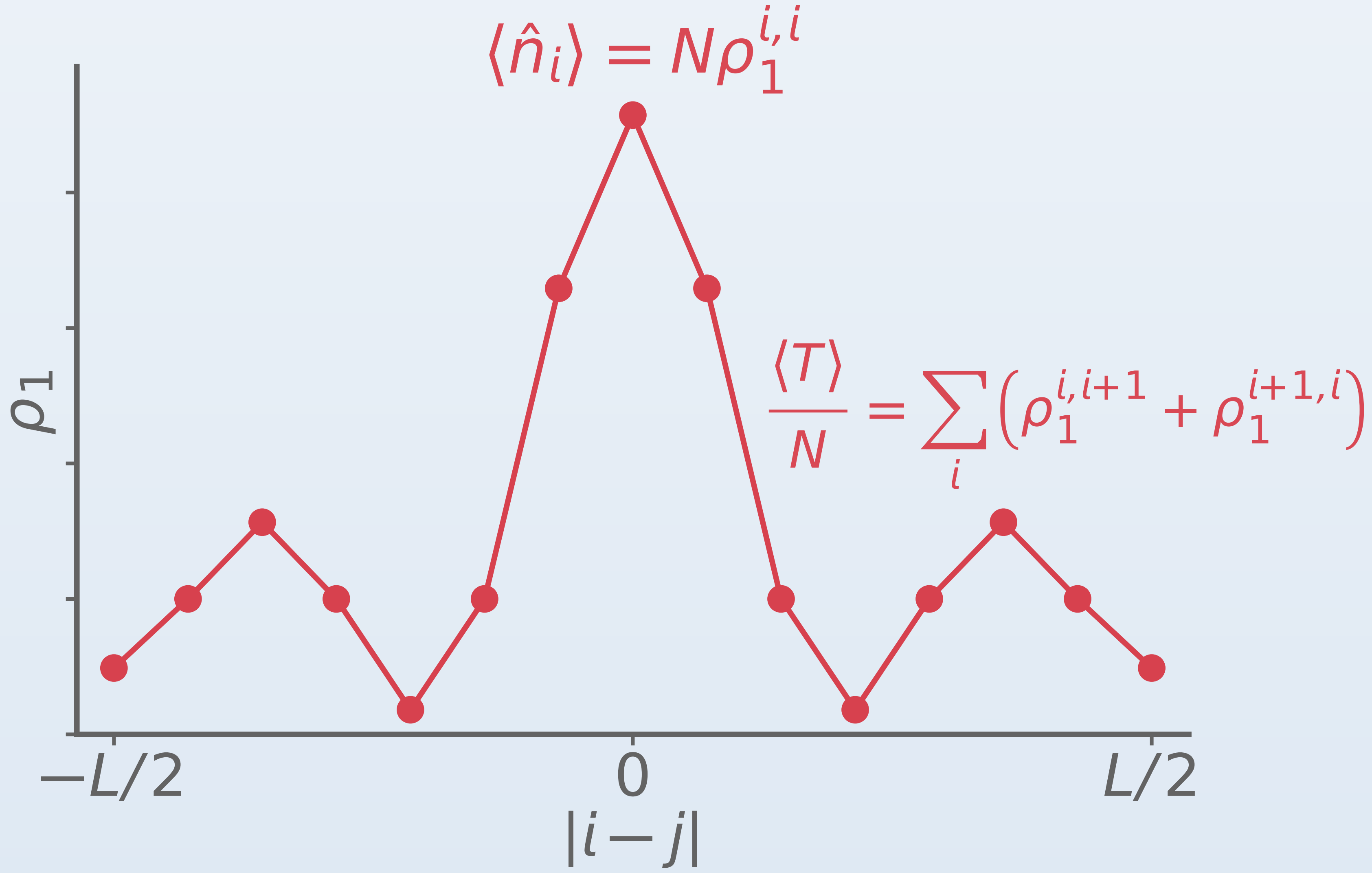
Example: free fermions

$$\rho_1^{i,j} = \frac{1}{NL}\frac{\sin\left(\frac{\pi N}{L}|i-j|\right)}{\sin\left(\frac{\pi}{L}|i-j|\right)}$$



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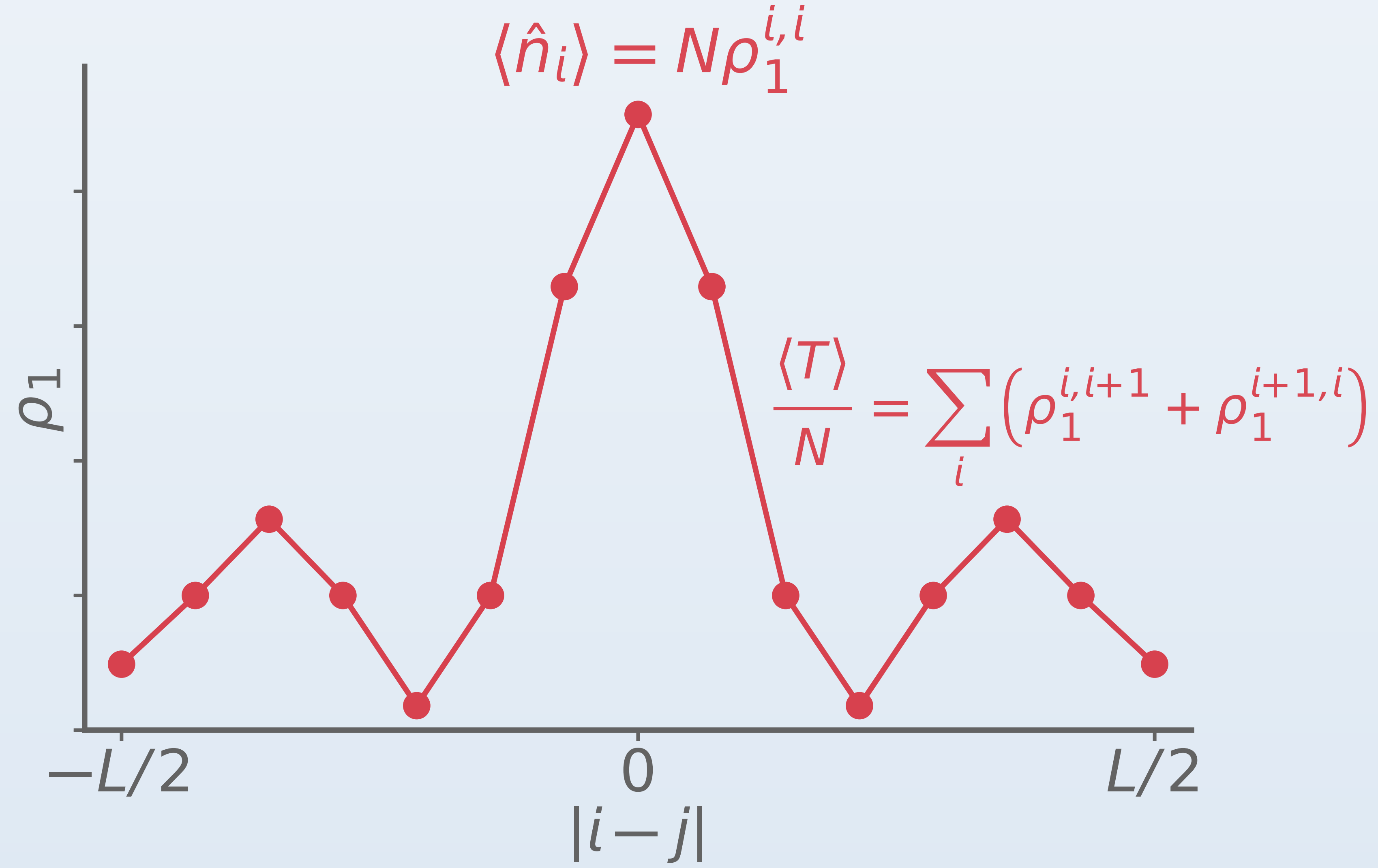
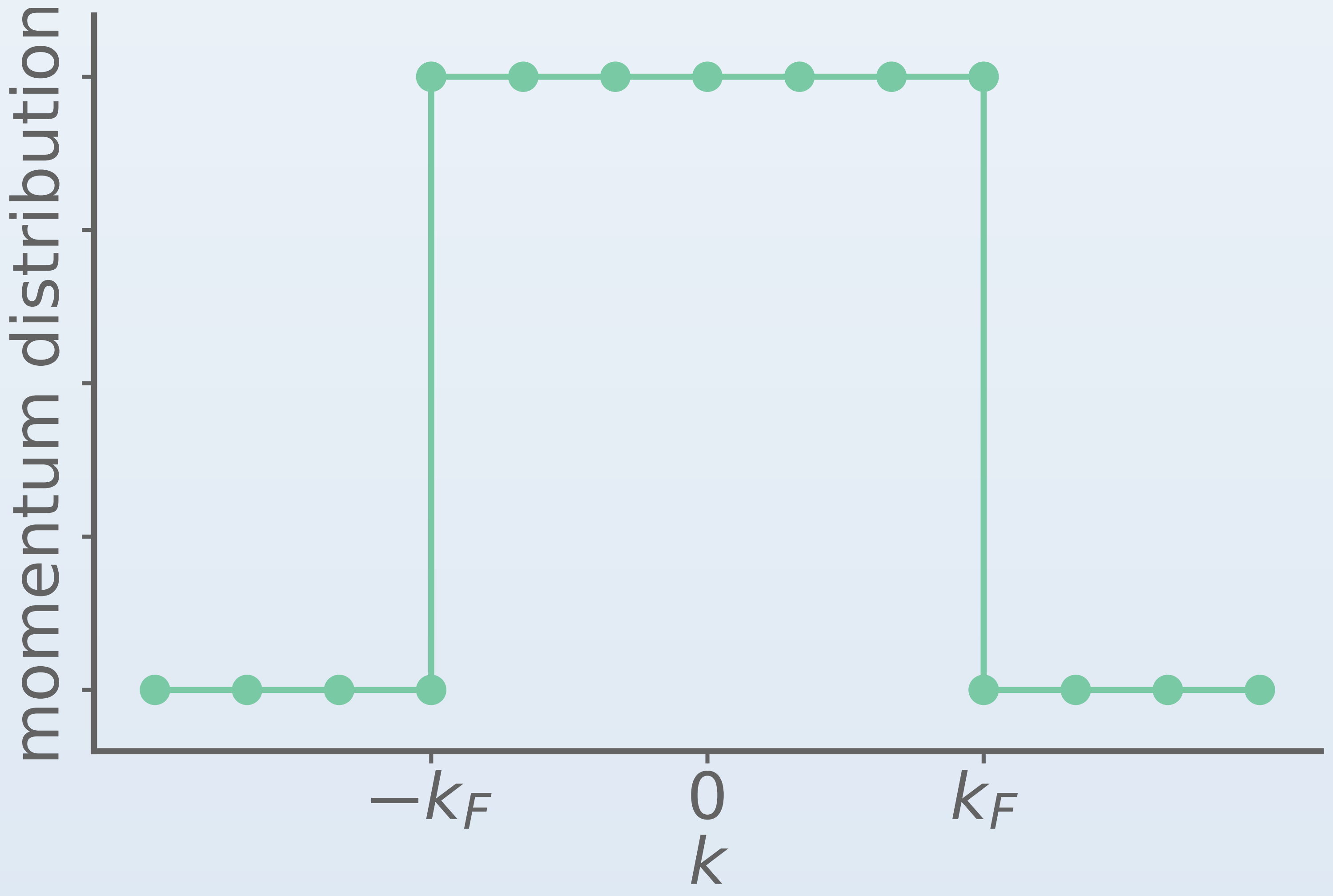
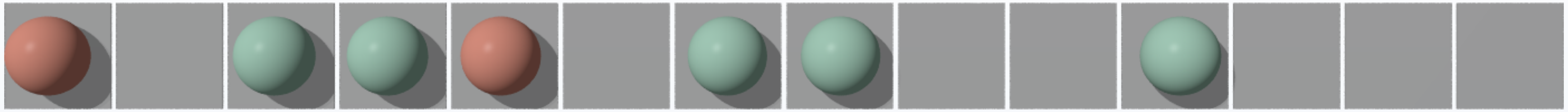




2-Particle Reduced Density Matrix



2-RDM: $\rho_2^{i_1,i_2;j_1,j_2} = \frac{(N-2)!}{N!} \langle\Psi| c_{i_1}^\dagger c_{i_2}^\dagger c_{j_1} c_{j_2} |\Psi\rangle$

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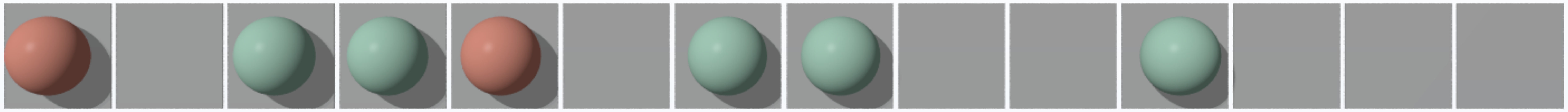

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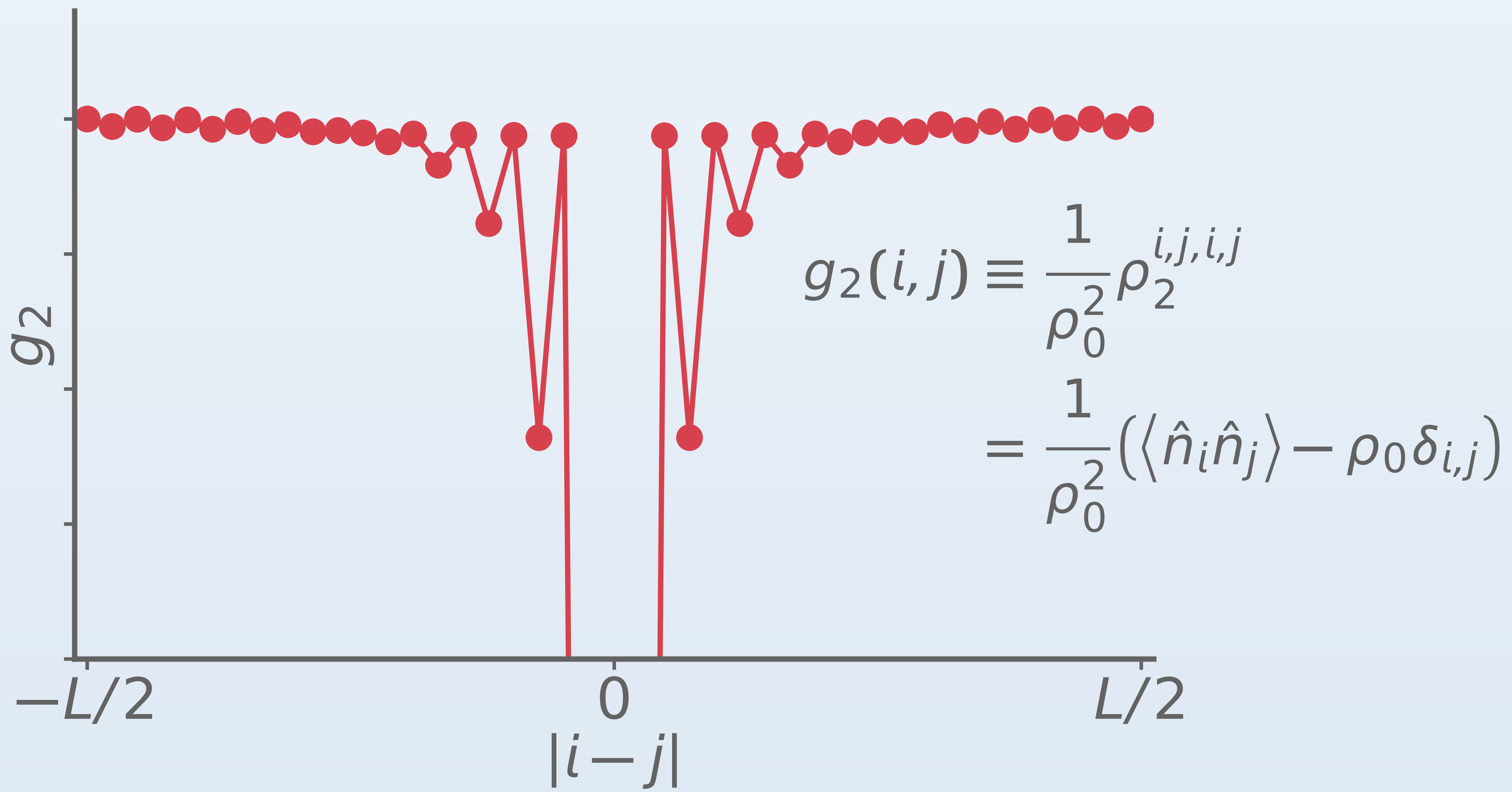
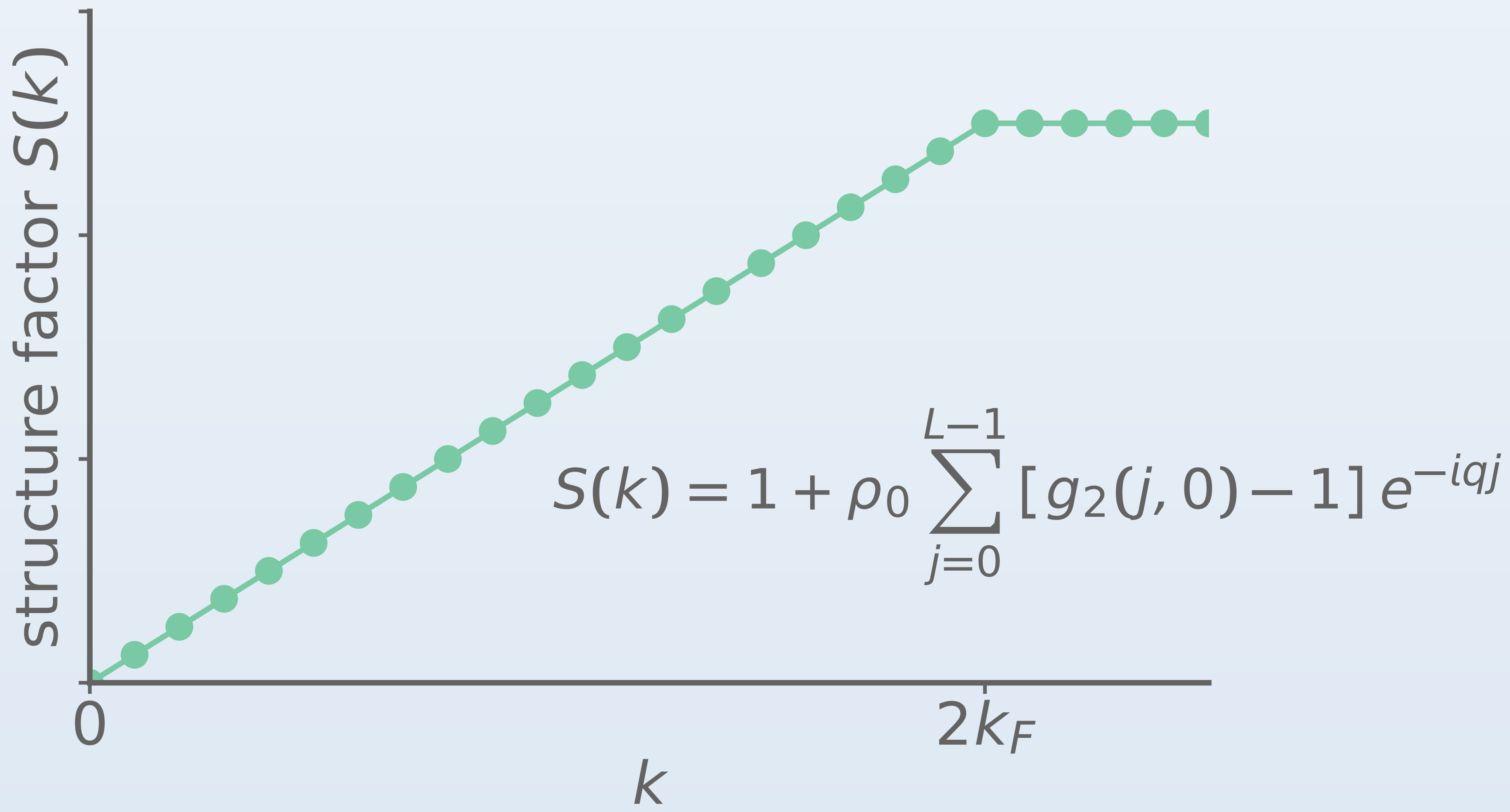
Example: free fermions - diagonal elements

pair correlation function



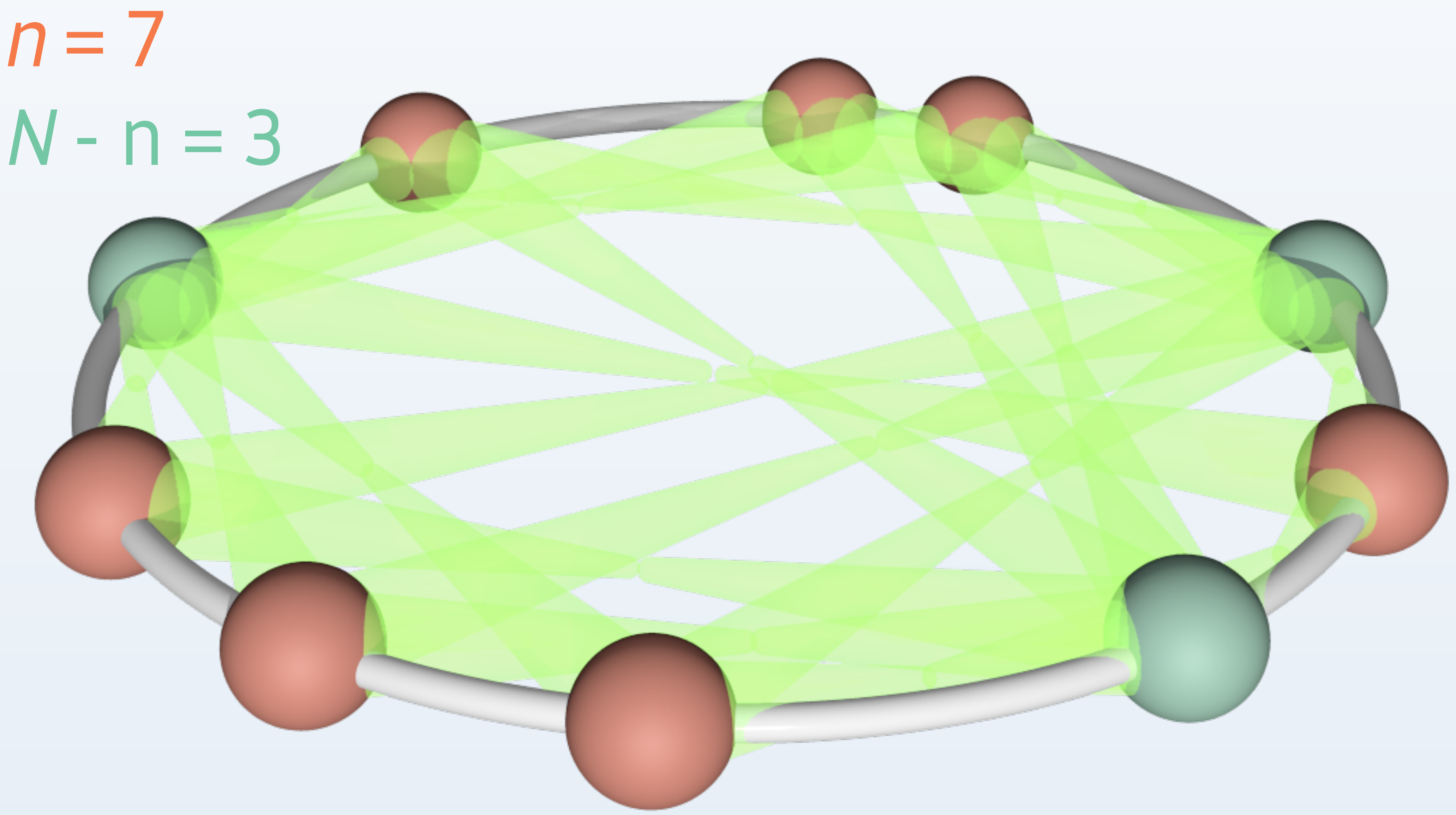
structure factor



Particle Partition Entanglement

Non-classical information encoded **non-locally** in the *n*-particle state of a system quantified by von Neumann entropy of the *n*-RDM:

$$S(n) = -\mathrm{Tr}\,\rho_n \ln \rho_n$$



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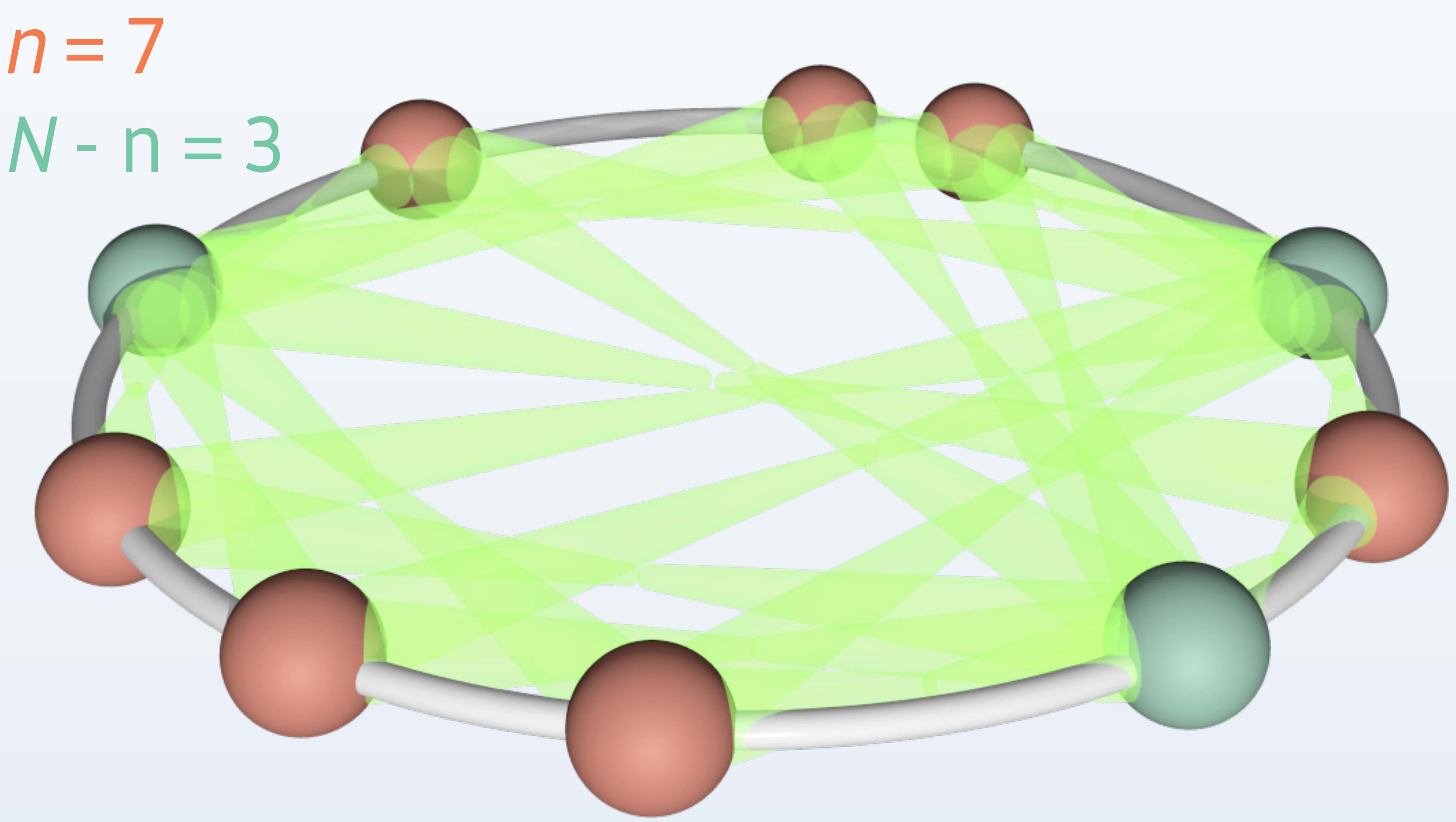

Distinct/complementary to conventionally measured mode entanglement

- no imposed external length scale
- independent of modes
- strongly dependent on interactions
- sensitive to particle statistics at leading order!

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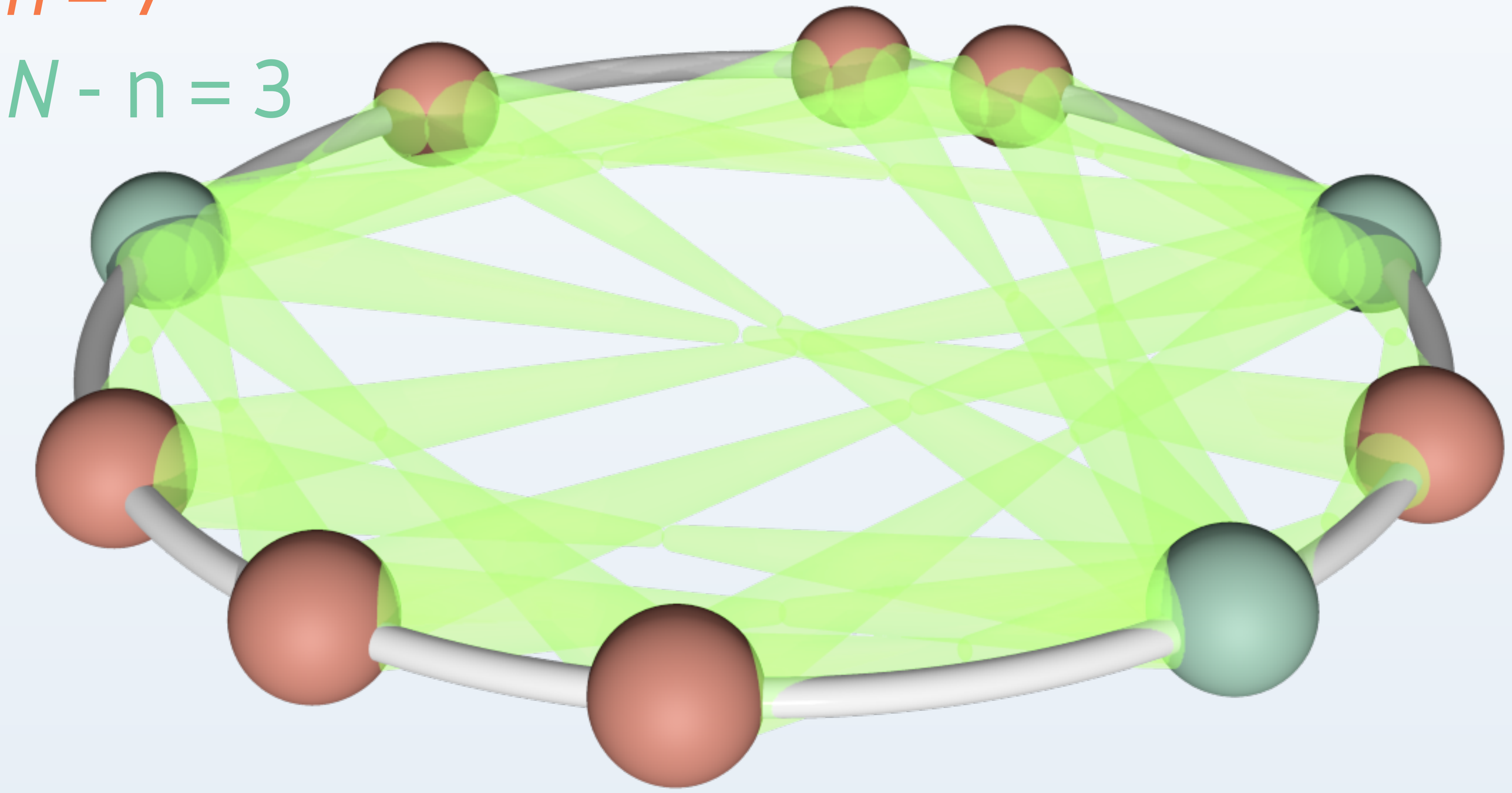

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non-interacting fermions: $S(n) = \ln \binom{N}{n}$

non-interacting bosons: $S(n) = 0$

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Lots to Learn About Particle Entanglement

monotonicity: $S(\rho_n) \leq S(\rho_{n+1})$ for $1 \leq n \leq N/2 - 1$

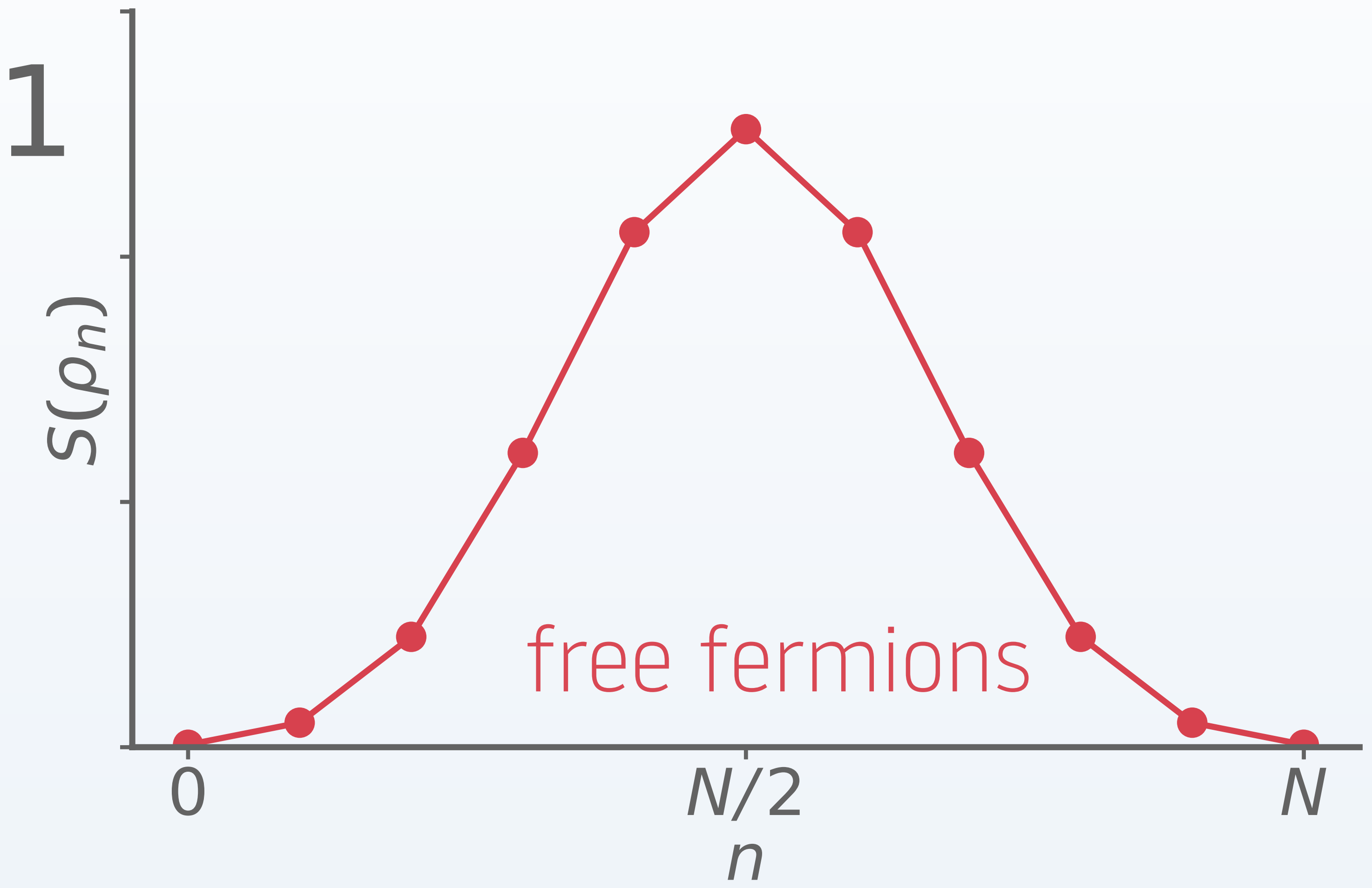
reflection: $S(\rho_n) = S(\rho_{N-n})$

concavity: $S(\rho_n) \geq [S(\rho_{n+1}) + S(\rho_{n-1})]/2$

bounds for fermions:

$$\ln N \leq S(\rho_1) \leq \ln L$$

$$2S(\rho_1) - S(\rho_2) \geq \ln 2 - \ln\left(1 - e^{-S(\rho_1)}\right)$$



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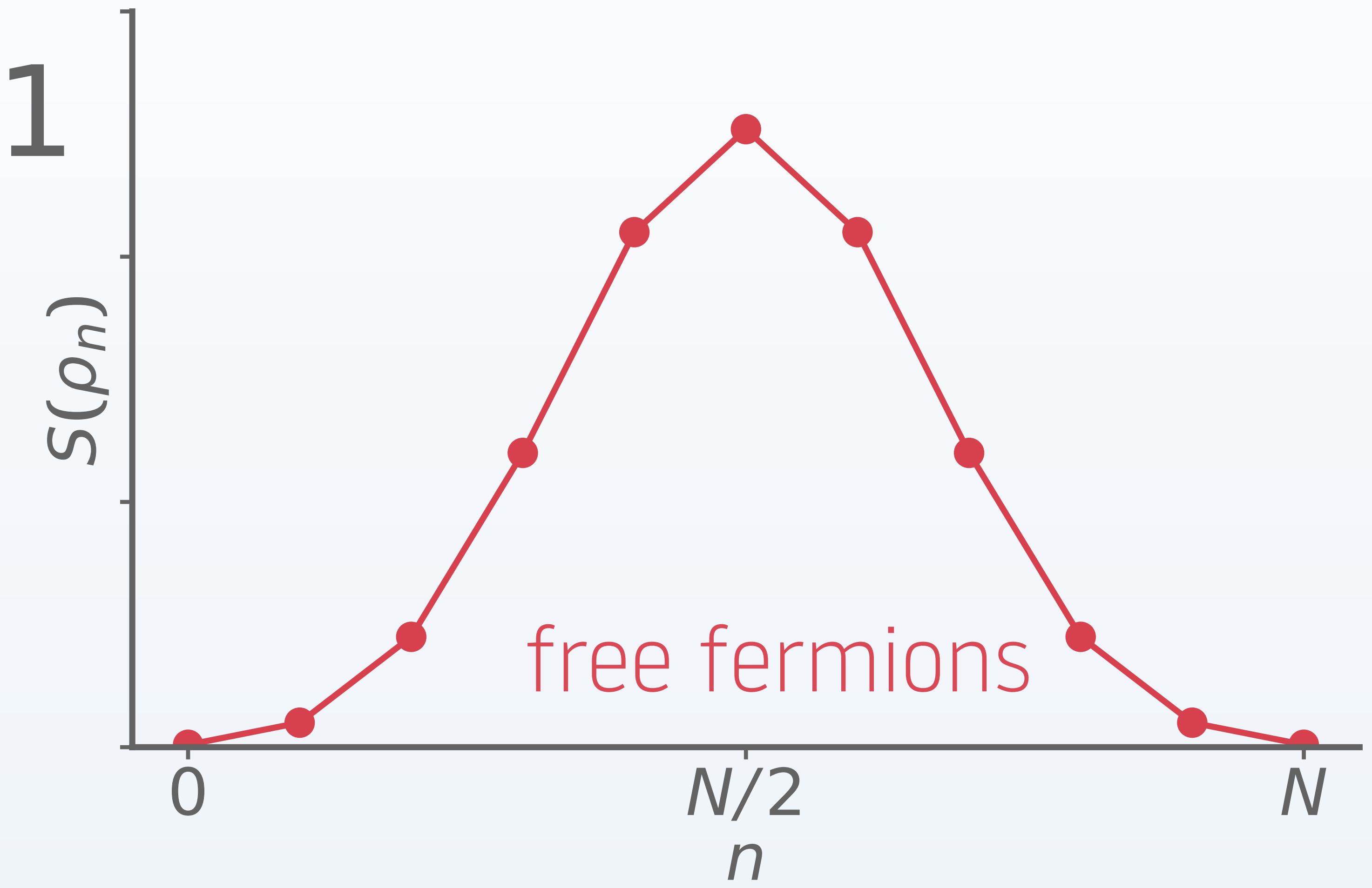

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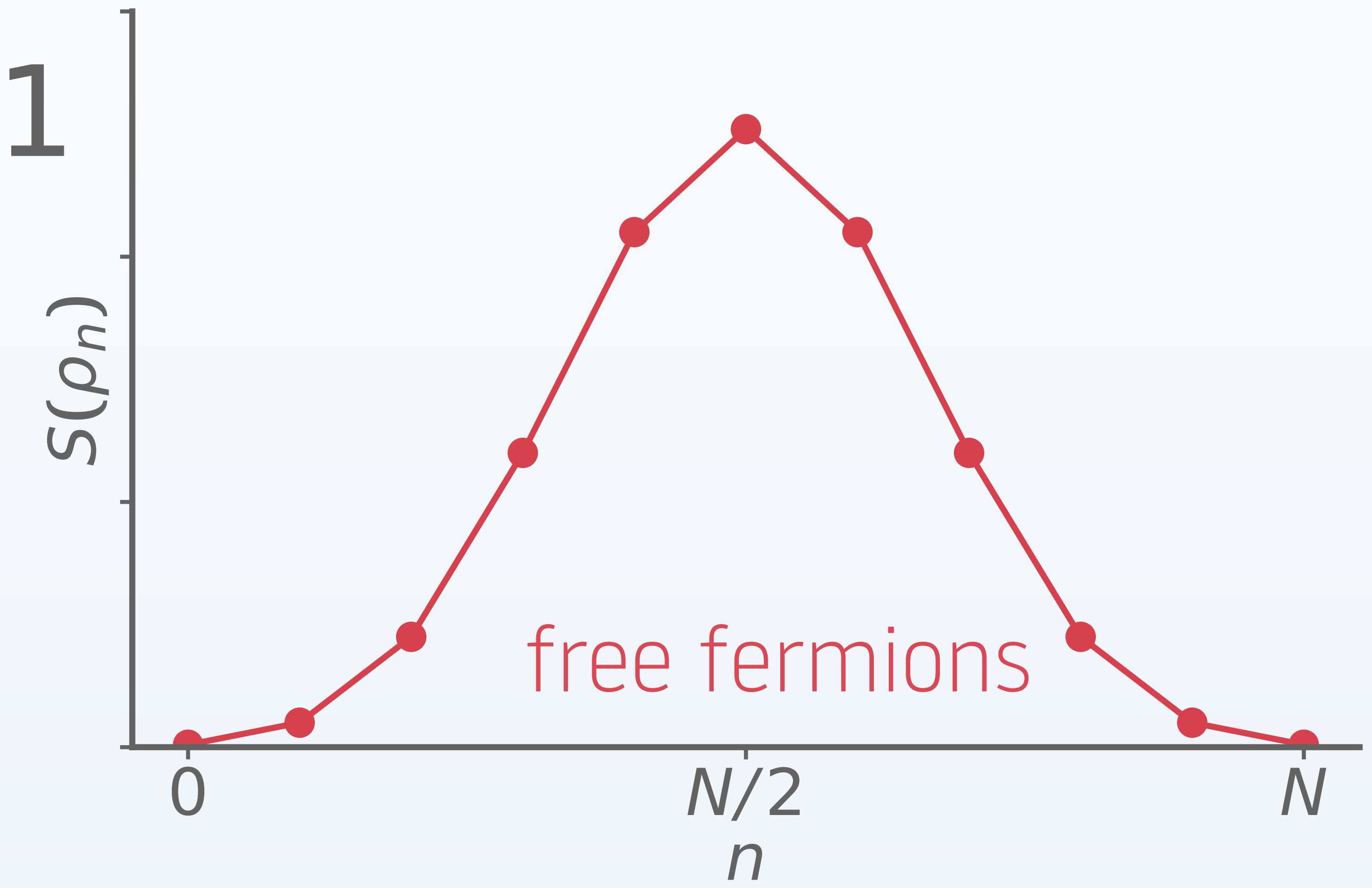

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Can a system of interacting fermions be ***less entangled*** than free fermions?

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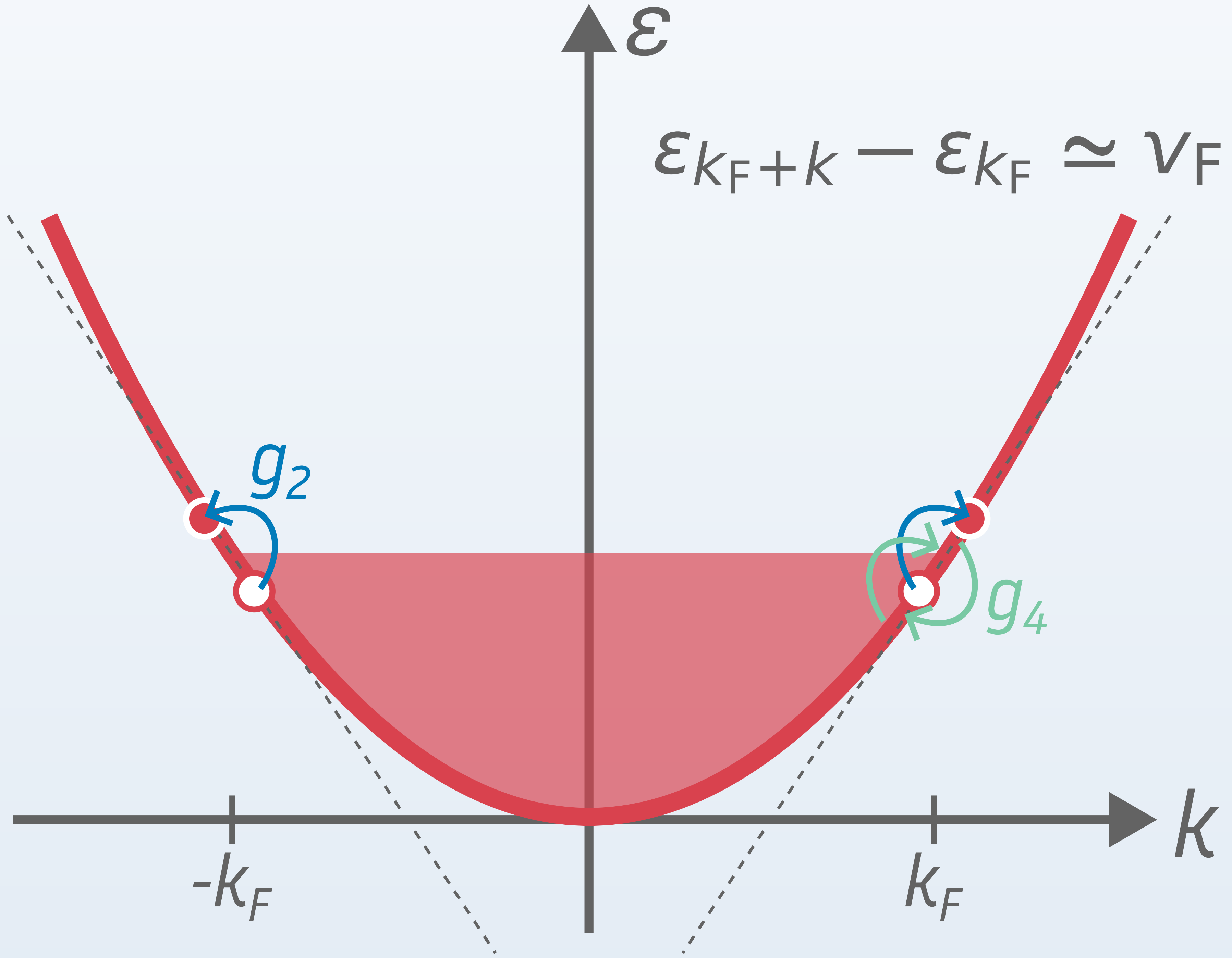
How can we compute n-RDMs for finite sized strongly interacting Fermi systems?

Tomonaga-Luttinger Model

Interacting fermions in 1D with forward scattering and a linearized dispersion

$$H = -\frac{1}{2M}\int dx\, \Psi^\dagger(x)\nabla_x^2\Psi(x) + \int dx' \int dx\, \varrho(x')V(x'-x)\varrho(x) \longleftarrow \varrho(x) = \Psi^\dagger(x)\Psi(x)$$



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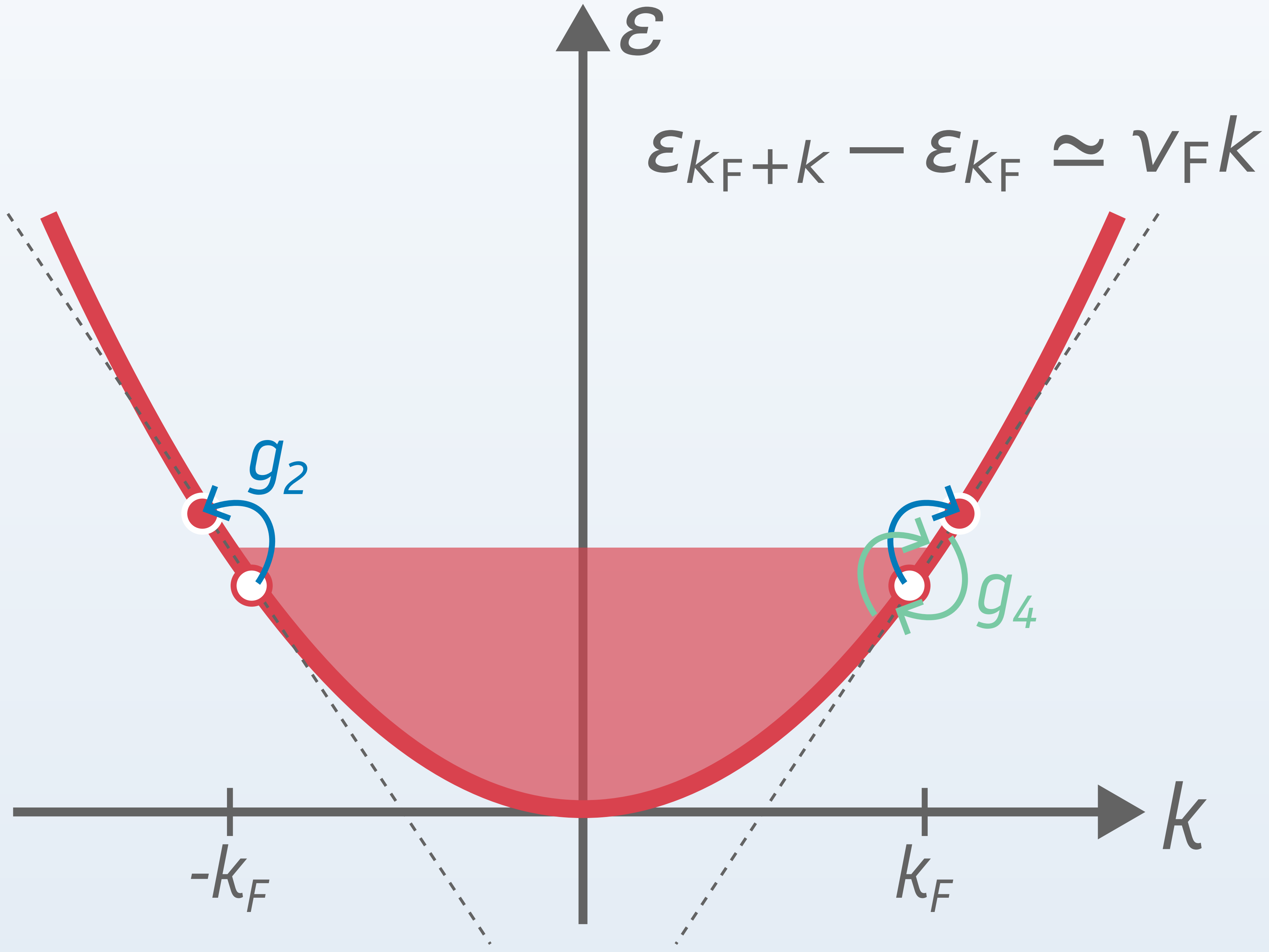

Exactly solvable via bosonization!

$$\Psi(x) = e^{ik_F x}\Psi_L(x) + e^{-ik_F x}\Psi_R(x)$$

$$\Psi_\alpha(x) = \frac{\chi_\alpha}{\sqrt{2\pi\eta}} e^{i(\varphi_{0,\alpha} + \alpha\frac{2\pi x}{L}N_\alpha)} e^{-i\phi_\alpha(x)}$$

$$\varrho_\alpha(x) = \frac{N_\alpha}{L} + \frac{\alpha}{2\pi}\partial_x\varphi_\alpha(x)$$

exponentiated boson field

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Constructive Bosonization

$$\phi_\alpha(x) = -\sum_{q>0} \sqrt{\frac{2\pi}{qL}} e^{-q\eta/2} \left[e^{\iota\alpha q x} b_{\alpha q} + e^{-\iota\alpha q x} b^\dagger_{\alpha q} \right]$$

← mode decomposition

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← mode decomposition

↓

$$H = \sum_{q\neq 0}[\omega_0(q) + m(q)]\, b^\dagger_q b_q + \frac{1}{2}\sum_{q\neq 0} g_2(q)\left(b_q b_{-q} + b^\dagger_q b^\dagger_{-q}\right)$$

$$= \text{z.m.} + \sum_{q\neq 0} v|q| a^\dagger_q a_q$$

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↓

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$$= \text{z.m.} + \sum_{q\neq 0} v|q| a^\dagger_q a_q \rightarrow H = \frac{v}{2\pi}\int \mathrm{d}x\left[\frac{1}{K}(\partial_x\phi)^2 + K(\partial_x\theta)^2\right]$$

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← mode decomposition

↓

$$H = \sum_{q\neq 0} \left[\omega_0(q) + m(q)\right] b^\dagger_q b_q + \frac{1}{2}\sum_{q\neq 0} g_2(q)\left(b_q b_{-q} + b^\dagger_q b^\dagger_{-q}\right)$$

$$= \text{z.m.} + \sum_{q\neq 0} v|q| a^\dagger_q a_q \rightarrow H = \frac{v}{2\pi}\int \mathrm{d}x\left[\frac{1}{K}(\partial_x\phi)^2 + K(\partial_x\theta)^2\right]$$

Can compute any fermonic correlation function via Bose-cumulant formula

$$\left\langle e^{i(\phi_\alpha(x)-\phi_\alpha(0))}\right\rangle = e^{-\frac{1}{2}\left\langle(\phi_\alpha(x)-\phi_\alpha(0))^2\right\rangle}$$

well known, see, e.g. Giamarchi Appendix C

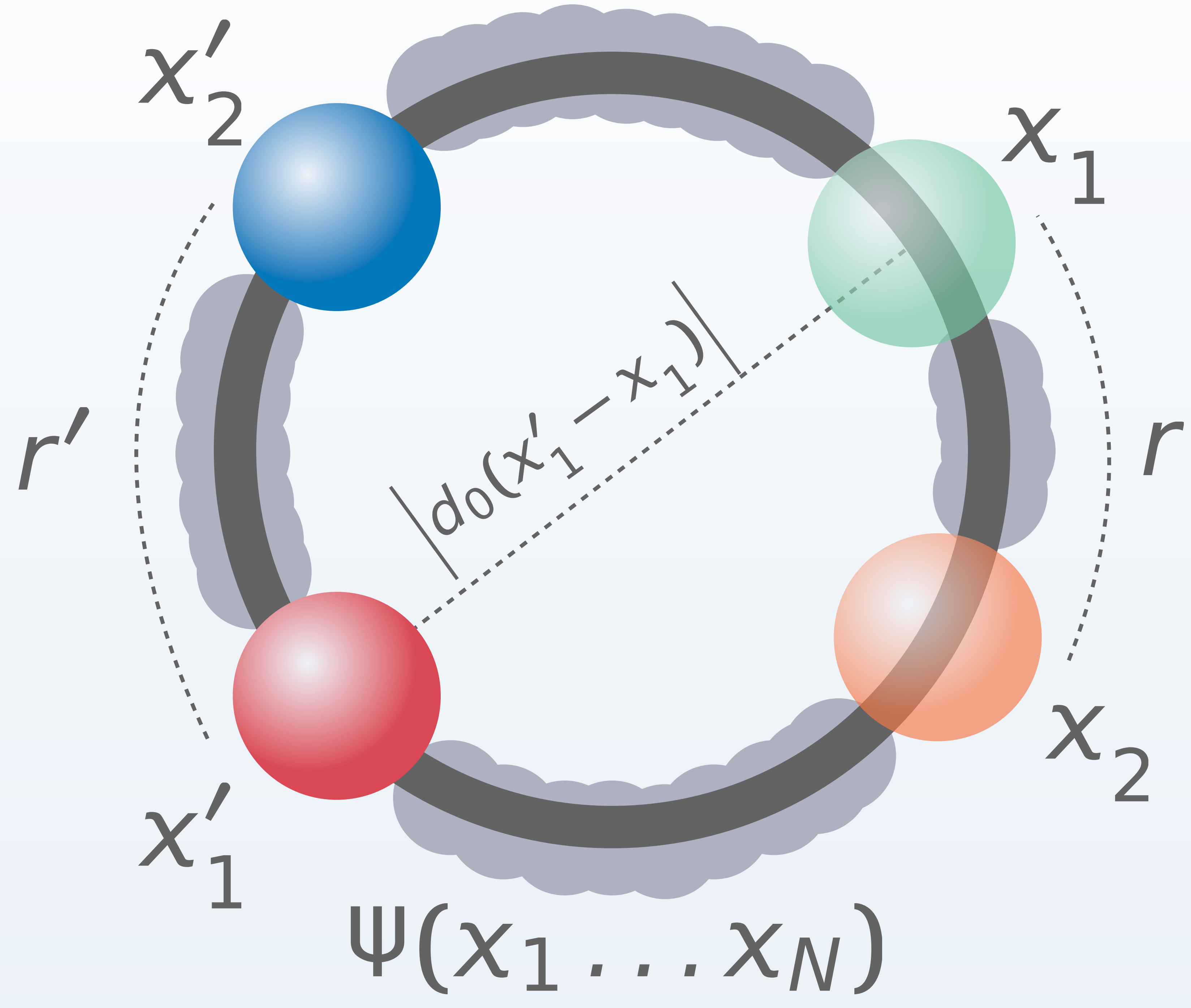
Vl. S. Dotsenko and V. A. Fateev, Nucl. Phys. B 240, 312 (1984)

or Tsvelik Chap. 26-27

$$e^{-\frac{1}{2}\sum_{i<j}\left[\left[-A_i A_j K - B_i B_j K^{-1}\right] F_1(r_i - r_j) + \left[A_i B_j + B_i A_j\right] F_2(r_i - r_j)\right]} \quad \text{(C.38)}$$

$$\langle A(1)\cdots A(4)\rangle = \left(\left|\frac{z_{13}z_{24}}{z_{12}z_{14}z_{23}z_{34}}\right|\right)^{4\Delta} G(x,\bar{x}) \quad (26.30)$$

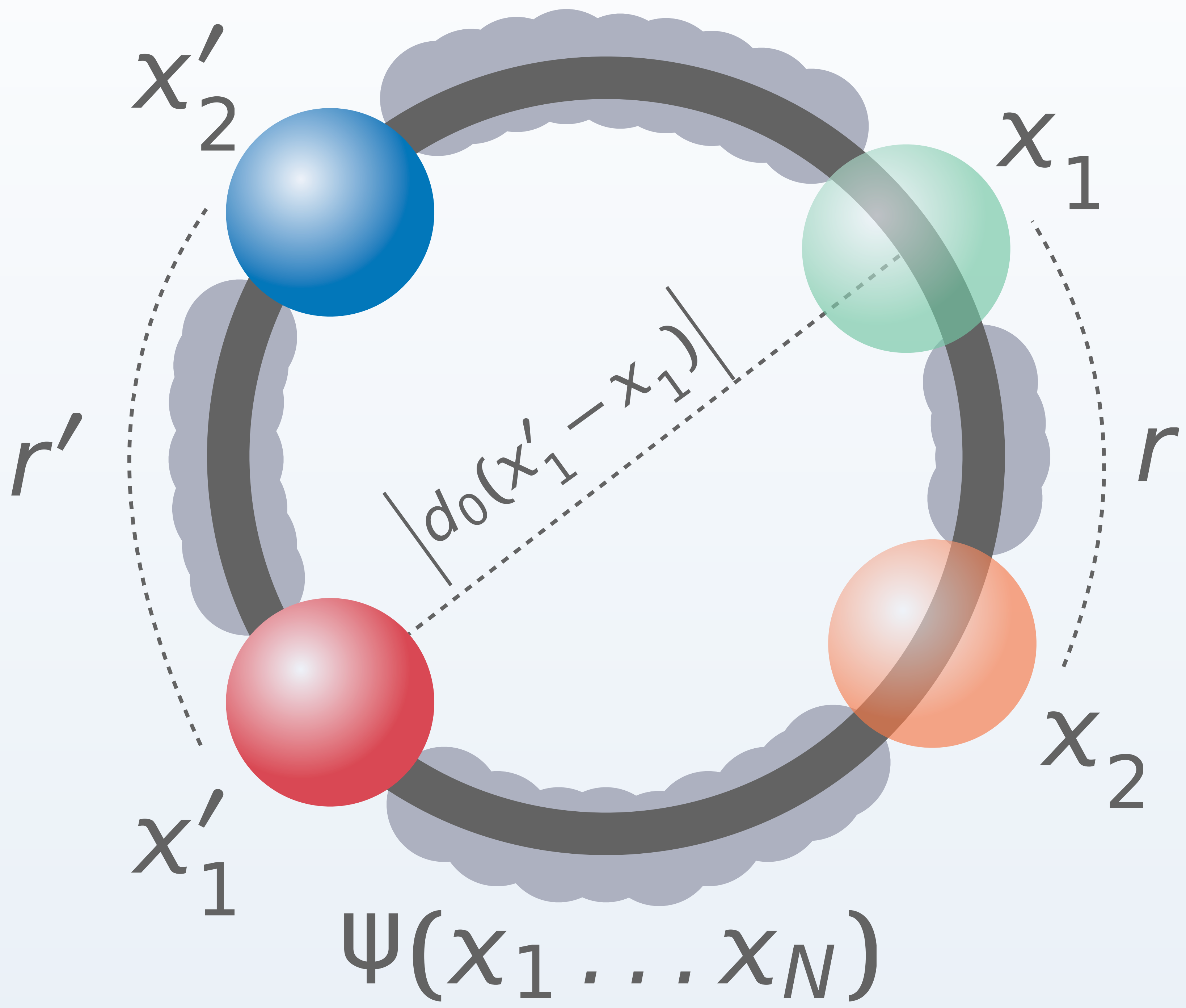
2-Particle Density Matrix: Bosonization



$$\rho_2(x'_2, x'_1, x_2, x_1) = \langle \Psi^\dagger(x'_2)\Psi^\dagger(x'_1)\Psi(x_1)\Psi(x_2)\rangle$$

Finite size L, periodic boundary conditions

2-Particle Density Matrix: Bosonization



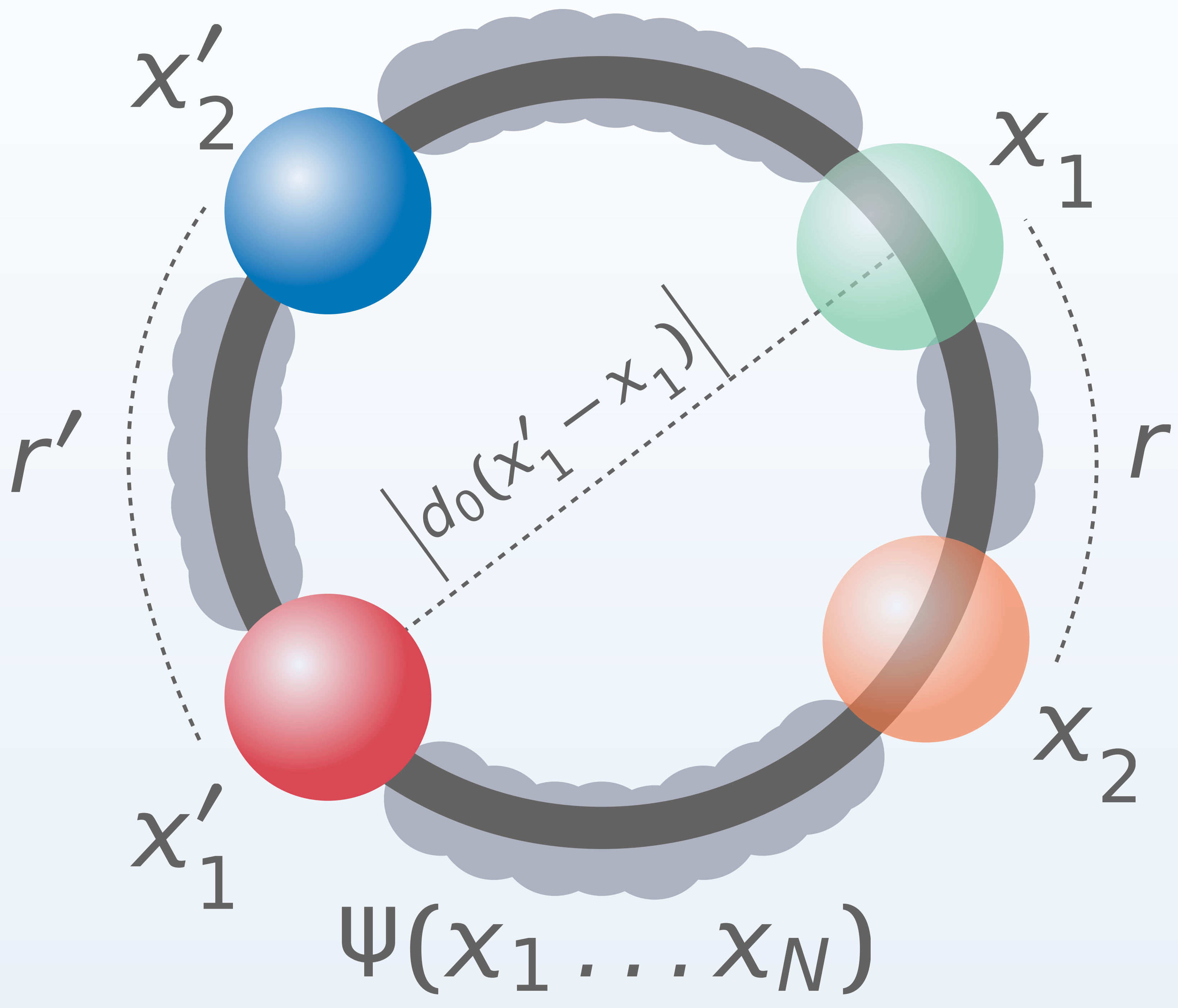
$$\rho_2(x_2', x_1', x_2, x_1) = \langle \Psi^\dagger(x_2')\Psi^\dagger(x_1')\Psi(x_1)\Psi(x_2)\rangle$$

Finite size L, periodic boundary conditions

$$\left\langle \Psi_\alpha^\dagger(x_2')\Psi_\alpha^\dagger(x_1')\Psi_\alpha(x_1)\Psi_\alpha(x_2)\right\rangle$$
$$\left\langle \Psi_\alpha^\dagger(x_2')\Psi_\beta^\dagger(x_1')\Psi_\alpha(x_1)\Psi_\beta(x_2)\right\rangle$$
$$\left\langle \Psi_\alpha^\dagger(x_2')\Psi_\beta^\dagger(x_1')\Psi_\beta(x_1)\Psi_\alpha(x_2)\right\rangle$$

6 surviving terms

2-Particle Density Matrix: Bosonization



$$\rho_2(x'_2, x'_1, x_2, x_1) = \langle \Psi^\dagger(x'_2)\Psi^\dagger(x'_1)\Psi(x_1)\Psi(x_2)\rangle$$

Finite size L, periodic boundary conditions

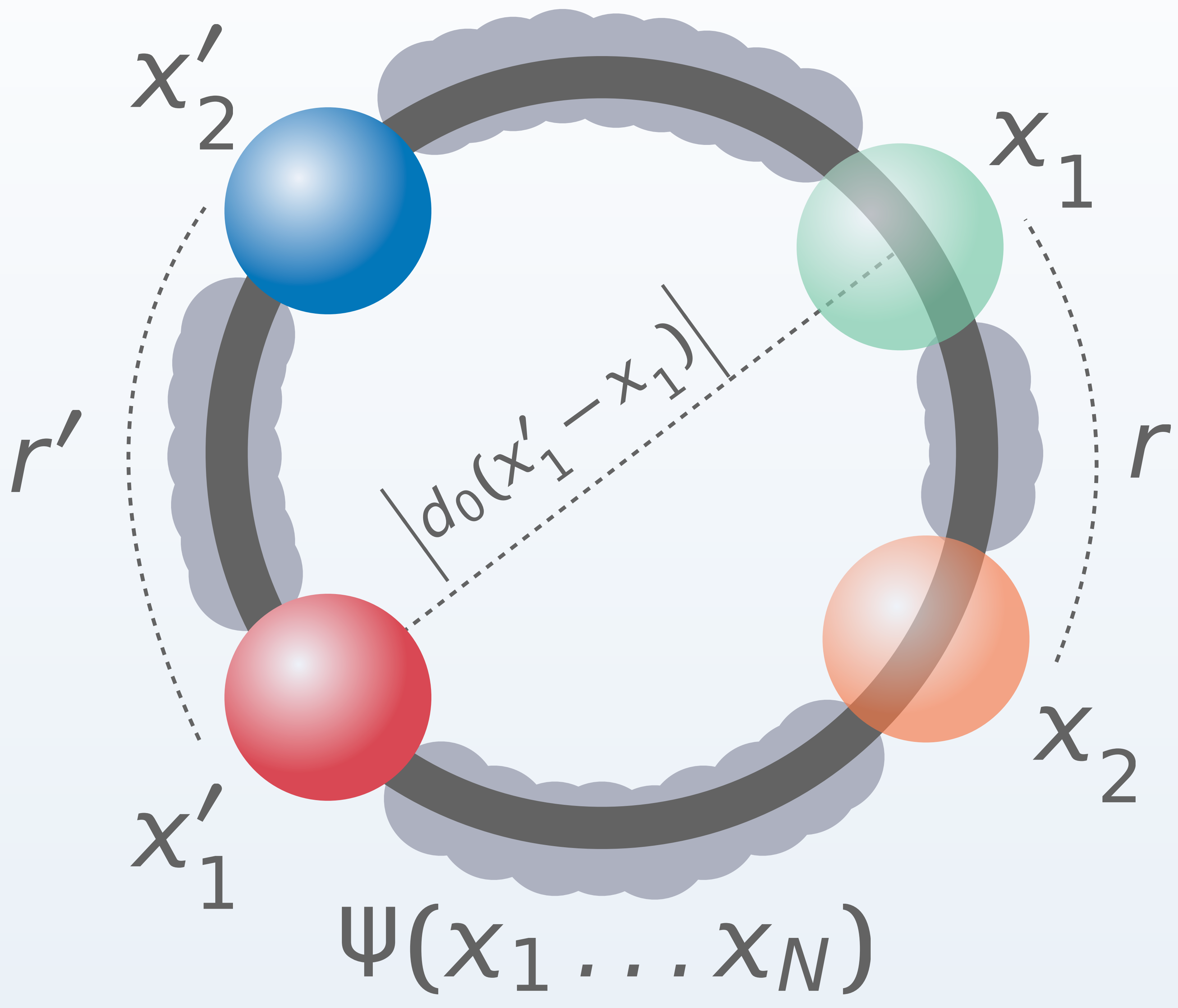
$$\left.\begin{matrix} \left\langle \Psi^\dagger_\alpha(x'_2)\Psi^\dagger_\alpha(x'_1)\Psi_\alpha(x_1)\Psi_\alpha(x_2)\right\rangle \\ \left\langle \Psi^\dagger_\alpha(x'_2)\Psi^\dagger_\beta(x'_1)\Psi_\alpha(x_1)\Psi_\beta(x_2)\right\rangle \\ \left\langle \Psi^\dagger_\alpha(x'_2)\Psi^\dagger_\beta(x'_1)\Psi_\beta(x_1)\Psi_\alpha(x_2)\right\rangle \end{matrix}\right\} \text{6 surviving terms}$$

$$d_\epsilon(x) = \frac{L}{\pi}\sin\left[\frac{\pi}{L}(x + i\epsilon)\right]$$

$$h_\epsilon(x, y) = d_\epsilon(x)d_\epsilon(y)$$

$$\begin{aligned} \langle \Psi^\dagger(x'_2)\Psi^\dagger(x'_1)\Psi(x_1)\Psi(x_2)\rangle = {} & \frac{\cos(k_F(x'_2+x'_1-x_2-x_1))}{2\pi^2}\left[\frac{h_0(x'_2-x'_1, x_2-x_1)}{h_0(x'_2-x_2, x'_1-x_1)h_0(x'_2-x_1, x'_1-x_2)}\right]\left|\frac{h_\epsilon(0,0)h_\epsilon(x'_2-x'_1, x_2-x_1)}{h_\epsilon(x'_2-x_2, x'_1-x_1)h_\epsilon(x'_2-x_1, x'_1-x_2)}\right|^{\gamma^2} \\ & + \frac{\cos(k_F(x'_2-x'_1-x_2+x_1))}{2\pi^2}\left[\frac{1}{h_0(x'_2-x_2, x'_1-x_1)}\right]\left|\frac{h_\epsilon(0,0)}{h_\epsilon(x'_2-x_2, x'_1-x_1)}\right|^{\gamma^2}\left|\frac{h_\epsilon(x'_2-x'_1, x_2-x_1)}{h_\epsilon(x'_2-x_1, x'_1-x_2)}\right|^{\lambda} \\ & - \frac{\cos(k_F(x'_2-x'_1+x_2-x_1))}{2\pi^2}\left[\frac{1}{h_0(x'_2-x_1, x'_1-x_2)}\right]\left|\frac{h_\epsilon(0,0)}{h_\epsilon(x'_2-x_1, x'_1-x_2)}\right|^{\gamma^2}\left|\frac{h_\epsilon(x'_2-x'_1, x_2-x_1)}{h_\epsilon(x'_2-x_2, x'_1-x_1)}\right|^{\lambda} \end{aligned}$$

2-Particle Density Matrix: Bosonization



$$\rho_2(x'_2, x'_1, x_2, x_1) = \langle \Psi^\dagger(x'_2)\Psi^\dagger(x'_1)\Psi(x_1)\Psi(x_2)\rangle$$

Finite size L, periodic boundary conditions

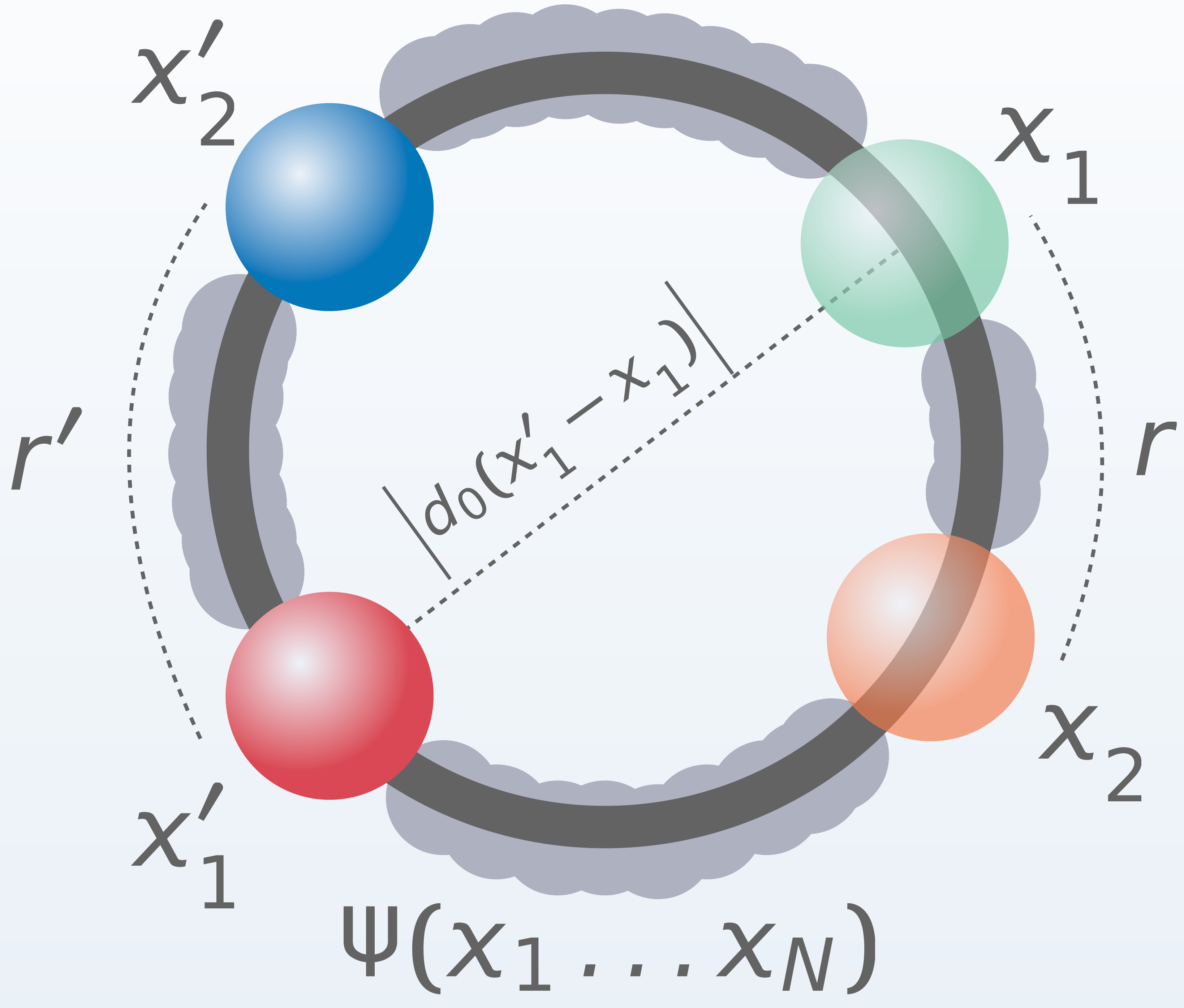
$$\left.\begin{array}{l}\left\langle \Psi^\dagger_\alpha(x'_2)\Psi^\dagger_\alpha(x'_1)\Psi_\alpha(x_1)\Psi_\alpha(x_2)\right\rangle \\ \left\langle \Psi^\dagger_\alpha(x'_2)\Psi^\dagger_\beta(x'_1)\Psi_\alpha(x_1)\Psi_\beta(x_2)\right\rangle \\ \left\langle \Psi^\dagger_\alpha(x'_2)\Psi^\dagger_\beta(x'_1)\Psi_\beta(x_1)\Psi_\alpha(x_2)\right\rangle \end{array}\right\} \text{6 surviving terms}$$

$$d_\epsilon(x) = \frac{L}{\pi}\sin\left[\frac{\pi}{L}(x + i\epsilon)\right]$$

$$h_\epsilon(x, y) = d_\epsilon(x)d_\epsilon(y)$$

$$\begin{aligned}\langle \Psi^\dagger(x'_2)\Psi^\dagger(x'_1)\Psi(x_1)\Psi(x_2)\rangle = {} & \frac{\cos\left(k_F(x'_2 + x'_1 - x_2 - x_1)\right)}{2\pi^2}\left[\frac{h_0(x'_2 - x'_1, x_2 - x_1)}{h_0(x'_2 - x_2, x'_1 - x_1)h_0(x'_2 - x_1, x'_1 - x_2)}\right]\left|\frac{h_\epsilon(0,0)h_\epsilon(x'_2 - x'_1, x_2 - x_1)}{h_\epsilon(x'_2 - x_2, x'_1 - x_1)h_\epsilon(x'_2 - x_1, x'_1 - x_2)}\right|^{\gamma^2} \\ & + \frac{\cos\left(k_F(x'_2 - x'_1 - x_2 + x_1)\right)}{2\pi^2}\left[\frac{1}{h_0(x'_2 - x_2, x'_1 - x_1)}\right]\left|\frac{h_\epsilon(0,0)}{h_\epsilon(x'_2 - x_2, x'_1 - x_1)}\right|^{\gamma^2}\left|\frac{h_\epsilon(x'_2 - x'_1, x_2 - x_1)}{h_\epsilon(x'_2 - x_1, x'_1 - x_2)}\right|^{\lambda} \\ & - \frac{\cos\left(k_F(x'_2 - x'_1 + x_2 - x_1)\right)}{2\pi^2}\left[\frac{1}{h_0(x'_2 - x_1, x'_1 - x_2)}\right]\left|\frac{h_\epsilon(0,0)}{h_\epsilon(x'_2 - x_1, x'_1 - x_2)}\right|^{\gamma^2}\left|\frac{h_\epsilon(x'_2 - x'_1, x_2 - x_1)}{h_\epsilon(x'_2 - x_2, x'_1 - x_1)}\right|^{\lambda}\end{aligned}$$

2-Particle Density Matrix: Bosonization



$$\rho_2(x_2', x_1', x_2, x_1) = \langle \Psi^\dagger(x_2')\Psi^\dagger(x_1')\Psi(x_1)\Psi(x_2)\rangle$$

Finite size L, periodic boundary conditions

$$\left\langle \Psi^\dagger_\alpha(x_2')\Psi^\dagger_\alpha(x_1')\Psi_\alpha(x_1)\Psi_\alpha(x_2)\right\rangle$$
$$\left\langle \Psi^\dagger_\alpha(x_2')\Psi^\dagger_\beta(x_1')\Psi_\alpha(x_1)\Psi_\beta(x_2)\right\rangle$$
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6 surviving terms

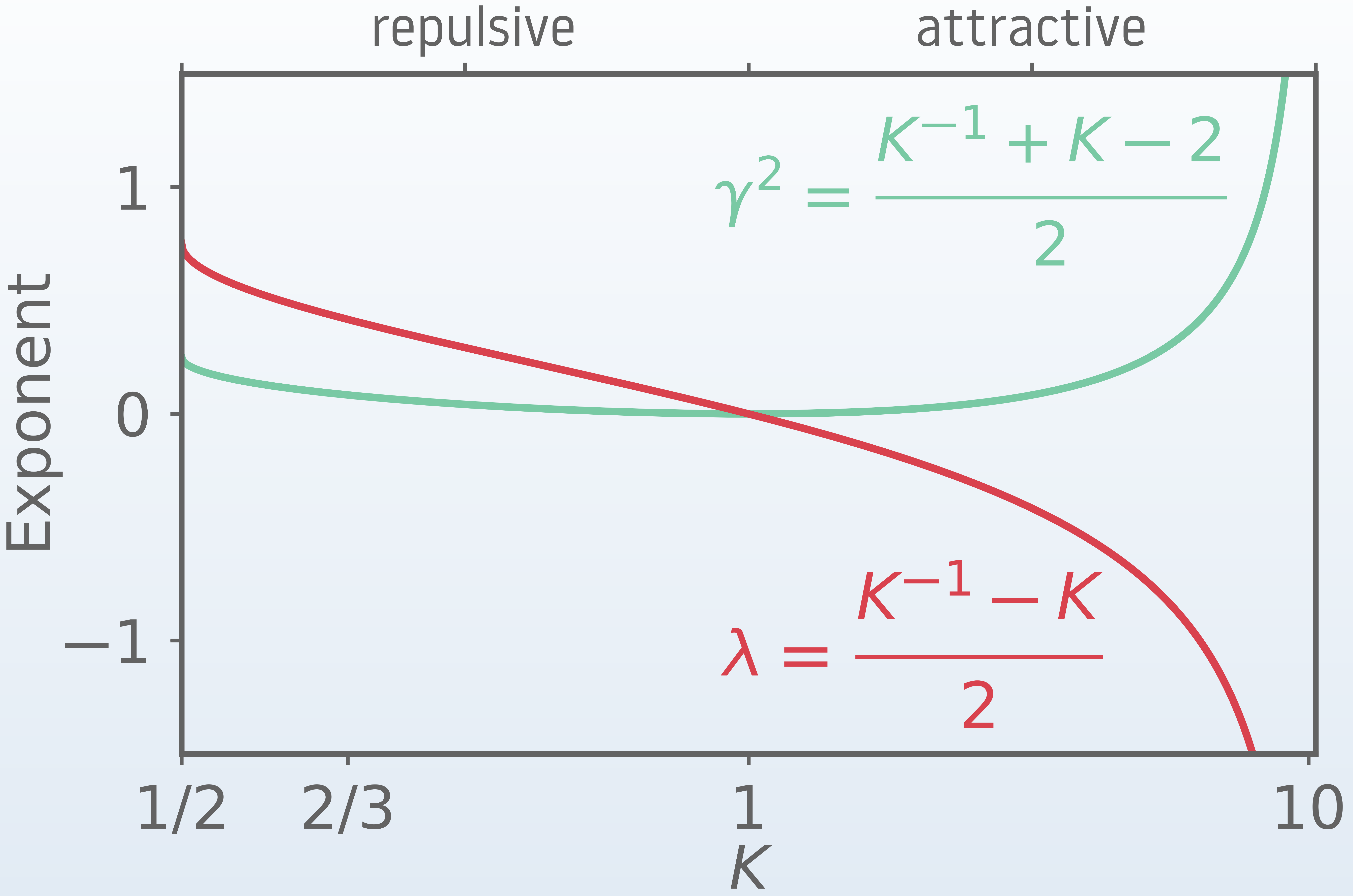
$$d_\epsilon(x) = \frac{L}{\pi}\sin\left[\frac{\pi}{L}(x+i\epsilon)\right]$$

$$h_\epsilon(x,y) = d_\epsilon(x)d_\epsilon(y)$$

$$\begin{aligned}
\langle \Psi^\dagger(x_2')\Psi^\dagger(x_1')\Psi(x_1)\Psi(x_2)\rangle = {} & \frac{\cos(k_F(x_2'+x_1'-x_2-x_1))}{2\pi^2}\left[\frac{h_0(x_2'-x_1', x_2-x_1)}{h_0(x_2'-x_2, x_1'-x_1)h_0(x_2'-x_1, x_1'-x_2)}\right]\left|\frac{h_\epsilon(0,0)h_\epsilon(x_2'-x_1', x_2-x_1)}{h_\epsilon(x_2'-x_2, x_1'-x_1)h_\epsilon(x_2'-x_1, x_1'-x_2)}\right|^{\gamma^2} \\
& + \frac{\cos(k_F(x_2'-x_1'-x_2+x_1))}{2\pi^2}\left[\frac{1}{h_0(x_2'-x_2, x_1'-x_1)}\right]\left|\frac{h_\epsilon(0,0)}{h_\epsilon(x_2'-x_2, x_1'-x_1)}\right|^{\gamma^2}\left|\frac{h_\epsilon(x_2'-x_1', x_2-x_1)}{h_\epsilon(x_2'-x_1, x_1'-x_2)}\right|^{\lambda} \\
& - \frac{\cos(k_F(x_2'-x_1'+x_2-x_1))}{2\pi^2}\left[\frac{1}{h_0(x_2'-x_1, x_1'-x_2)}\right]\left|\frac{h_\epsilon(0,0)}{h_\epsilon(x_2'-x_1, x_1'-x_2)}\right|^{\gamma^2}\left|\frac{h_\epsilon(x_2'-x_1', x_2-x_1)}{h_\epsilon(x_2'-x_2, x_1'-x_1)}\right|^{\lambda}
\end{aligned}$$

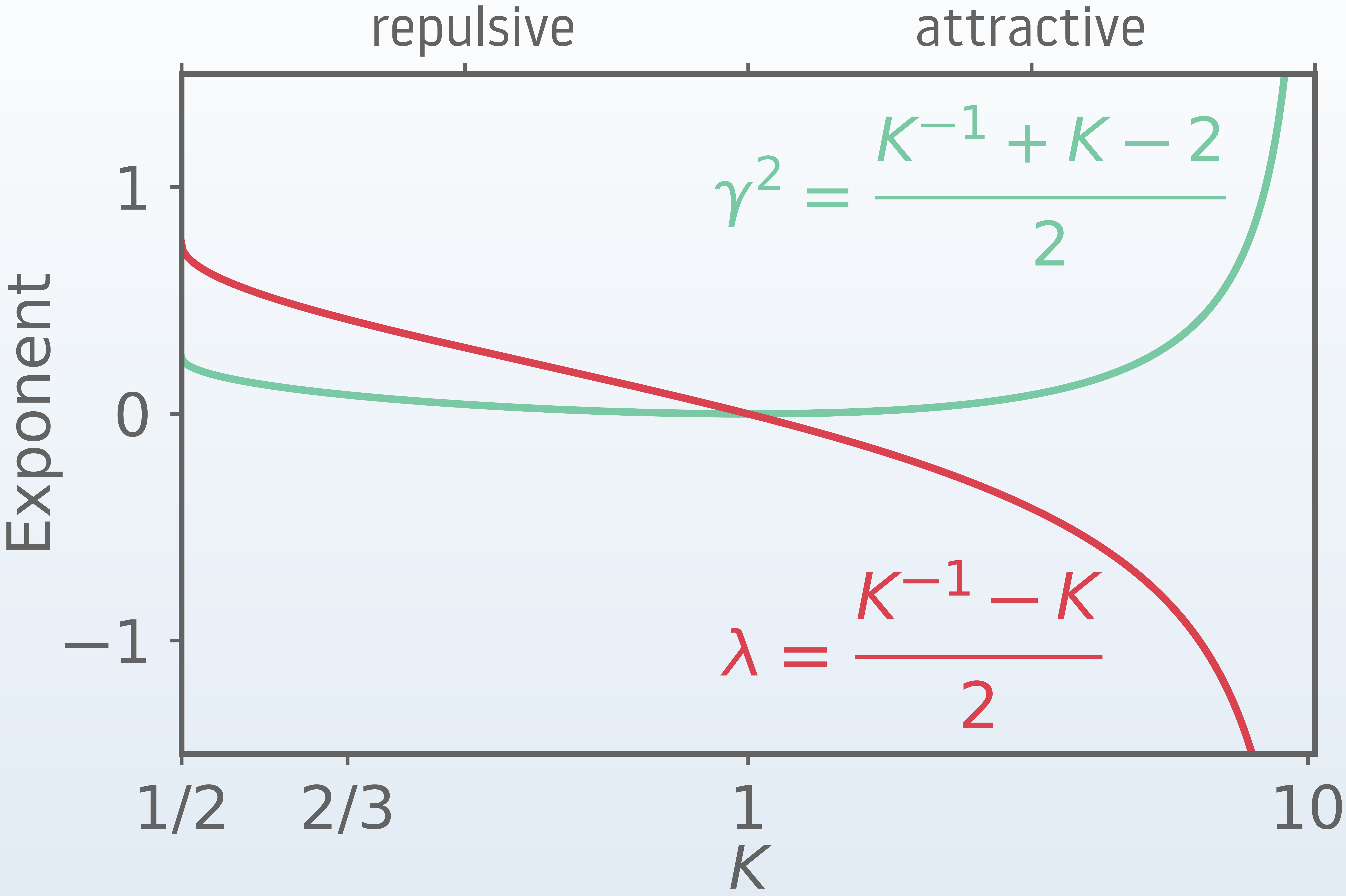
new!

2-RDM: Interaction Exponents



$$\rho_2(x_2', x_1', x_1, x_2)\Big|_{\substack{K=1 \\ \gamma^2=\lambda=0}} = \langle \Psi^\dagger(x_2')\Psi(x_2)\rangle_0 \langle \Psi^\dagger(x_1')\Psi(x_1)\rangle_0 - \langle \Psi^\dagger(x_2')\Psi(x_1)\rangle_0 \langle \Psi^\dagger(x_1')\Psi(x_2)\rangle_0$$

2-RDM: Interaction Exponents



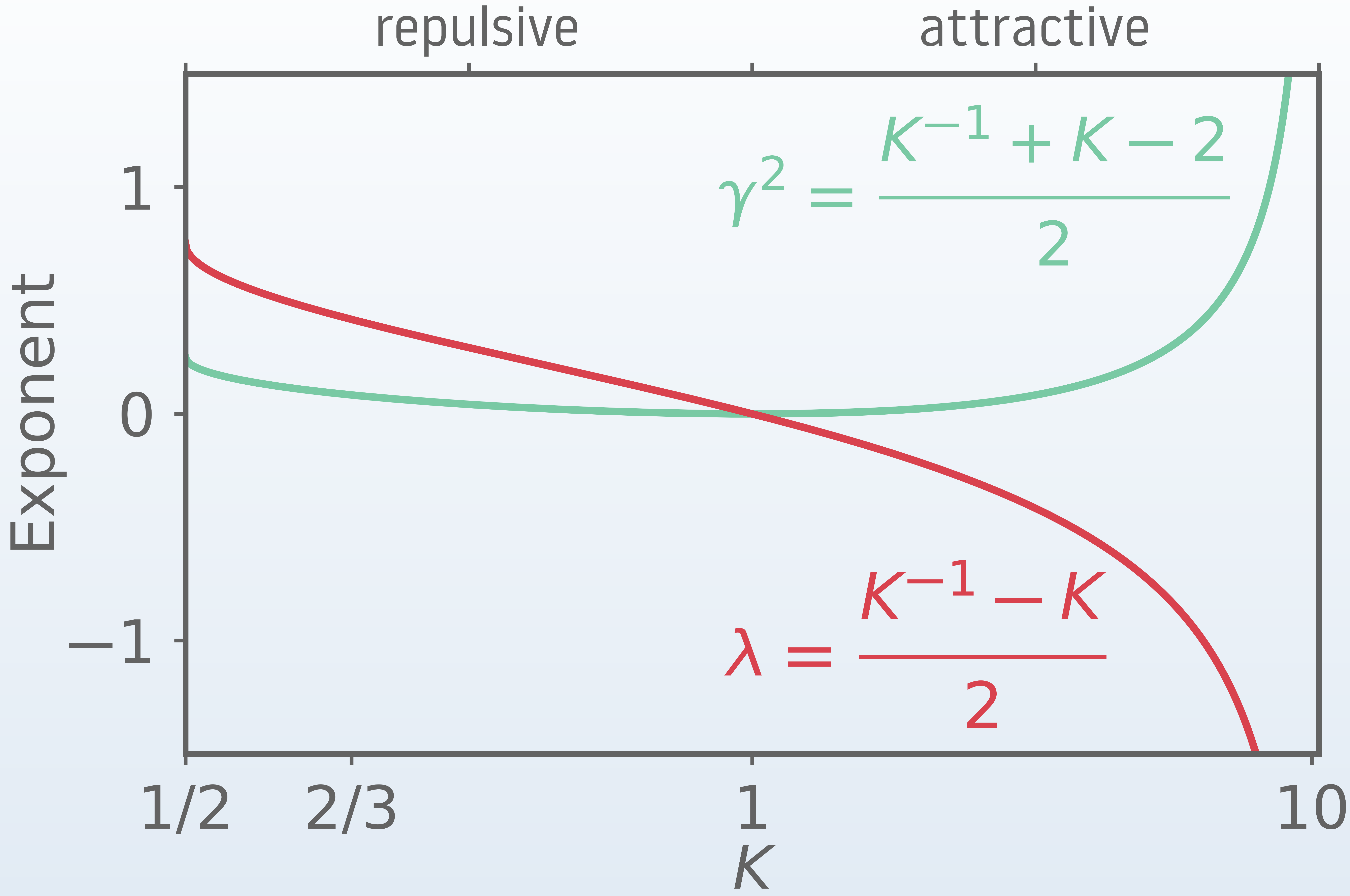
γ^2 appears in 1-RDM

$$\rho_1(x_1', x_1) = \rho_{1,\mathrm{FF}}(x_1', x_1) \left| \frac{\sin(\pi i \epsilon / L)}{d_\epsilon(x_1' - x_1)} \right|^{\gamma^2}$$

M. A. Cazalilla, Phys Rev Lett 97, 156403 (2006).
M. Thamm, H. Radhakrishnan, H. Barghathi, B. Rosenow, AD, Phys. Rev. B 106, 165116 (2022)

$$\rho_2(x_2', x_1', x_1, x_2)\Big|_{\substack{K=1 \\ \gamma^2=\lambda=0}} = \langle \Psi^\dagger(x_2')\Psi(x_2)\rangle_0 \langle \Psi^\dagger(x_1')\Psi(x_1)\rangle_0 - \langle \Psi^\dagger(x_2')\Psi(x_1)\rangle_0 \langle \Psi^\dagger(x_1')\Psi(x_2)\rangle_0$$

2-RDM: Interaction Exponents



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Explore **interaction effects** related to λ

$$\rho_2(x_2', x_1', x_1, x_2)\Big|_{\substack{K=1 \\ \gamma^2=\lambda=0}} = \langle \Psi^\dagger(x_2')\Psi(x_2)\rangle_0 \langle \Psi^\dagger(x_1')\Psi(x_1)\rangle_0 - \langle \Psi^\dagger(x_2')\Psi(x_1)\rangle_0 \langle \Psi^\dagger(x_1')\Psi(x_2)\rangle_0$$

2-RDM: Coordinates & Structure

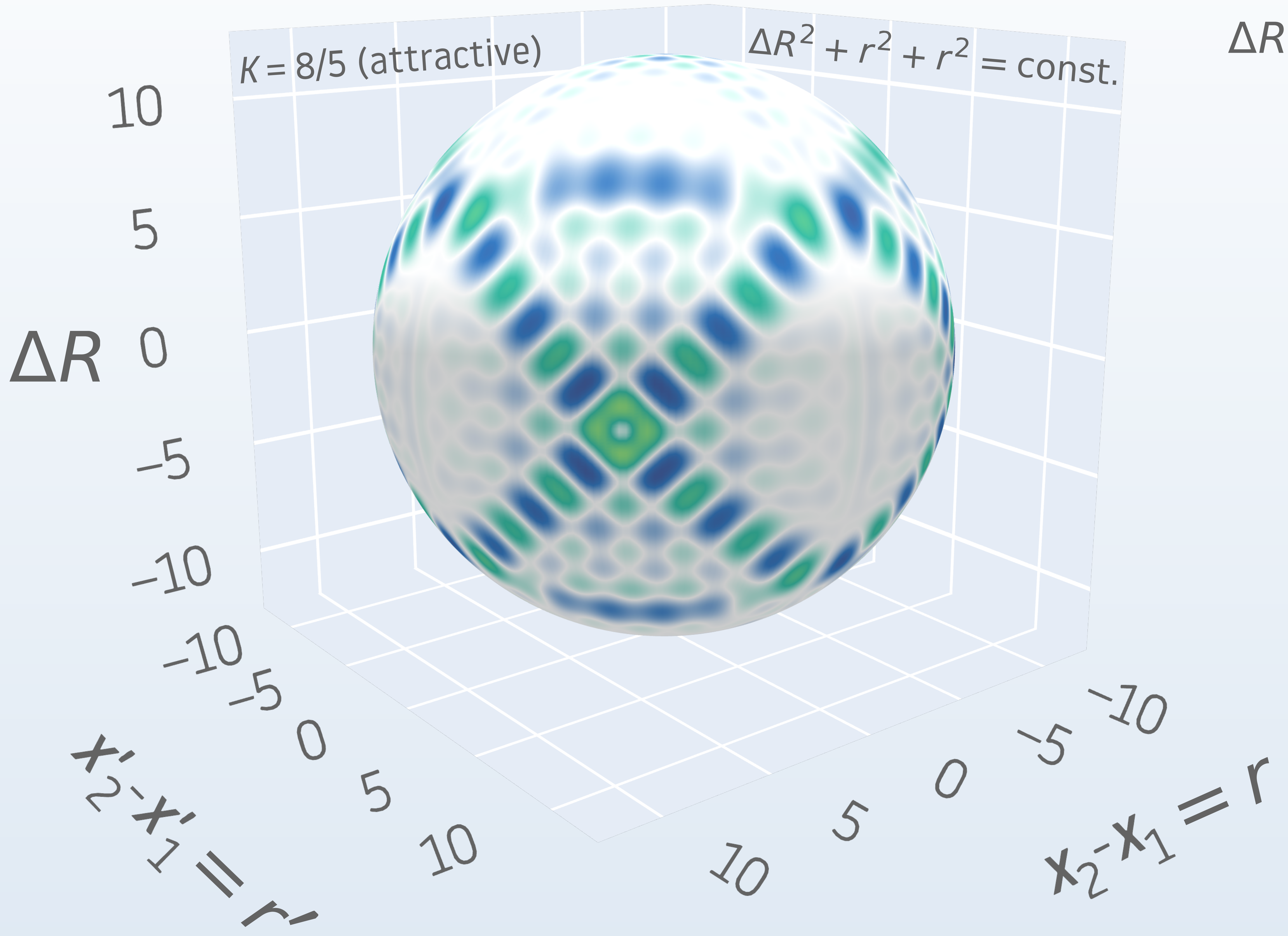
$$\rho_2(x_2', x_1', x_2, x_1) = \langle \Psi^\dagger(x_2')\Psi^\dagger(x_1')\Psi(x_1)\Psi(x_2)\rangle$$

$$\Sigma R \equiv R' + R = \tfrac{1}{2}(x_2' + x_1' + x_2 + x_1) = \text{const.}$$

$$\Delta R = R' - R = \tfrac{1}{2}(x_2' + x_1' - x_2 - x_1)$$

2-RDM: Coordinates & Structure

$$\rho_2(x_2', x_1', x_2, x_1) = \langle \Psi^\dagger(x_2')\Psi^\dagger(x_1')\Psi(x_1)\Psi(x_2)\rangle$$



$$\Sigma R \equiv R' + R = \frac{1}{2}(x_2' + x_1' + x_2 + x_1) = \text{const.}$$

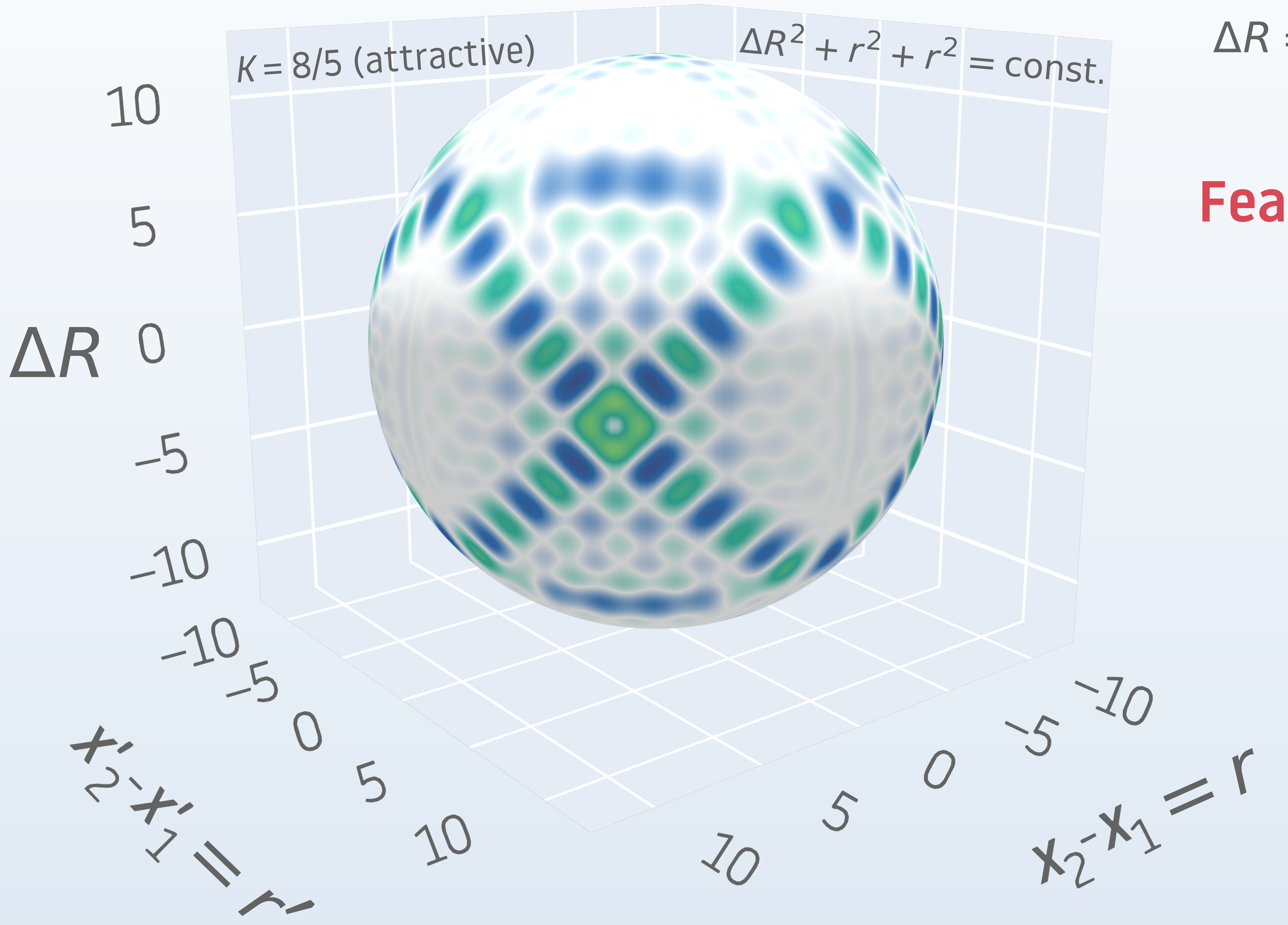
$$\Delta R = R' - R = \frac{1}{2}(x_2' + x_1' - x_2 - x_1)$$

2-RDM: Coordinates & Structure

$$\rho_2(x_2', x_1', x_2, x_1) = \langle \Psi^\dagger(x_2')\Psi^\dagger(x_1')\Psi(x_1)\Psi(x_2) \rangle$$

$$\Sigma R \equiv R' + R = \tfrac{1}{2}(x_2' + x_1' + x_2 + x_1) = \text{const.}$$

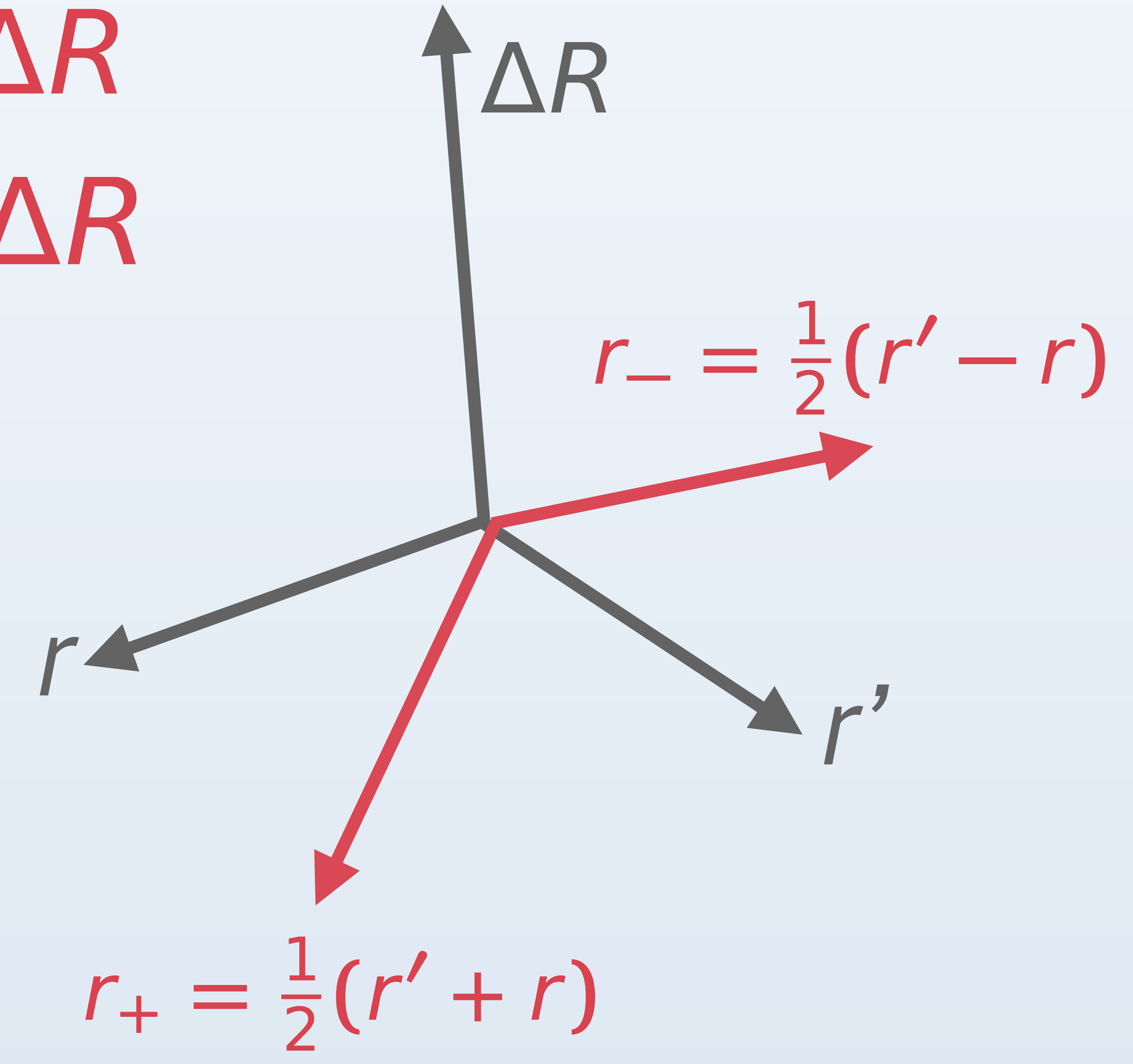
$$\Delta R = R' - R = \tfrac{1}{2}(x_2' + x_1' - x_2 - x_1)$$



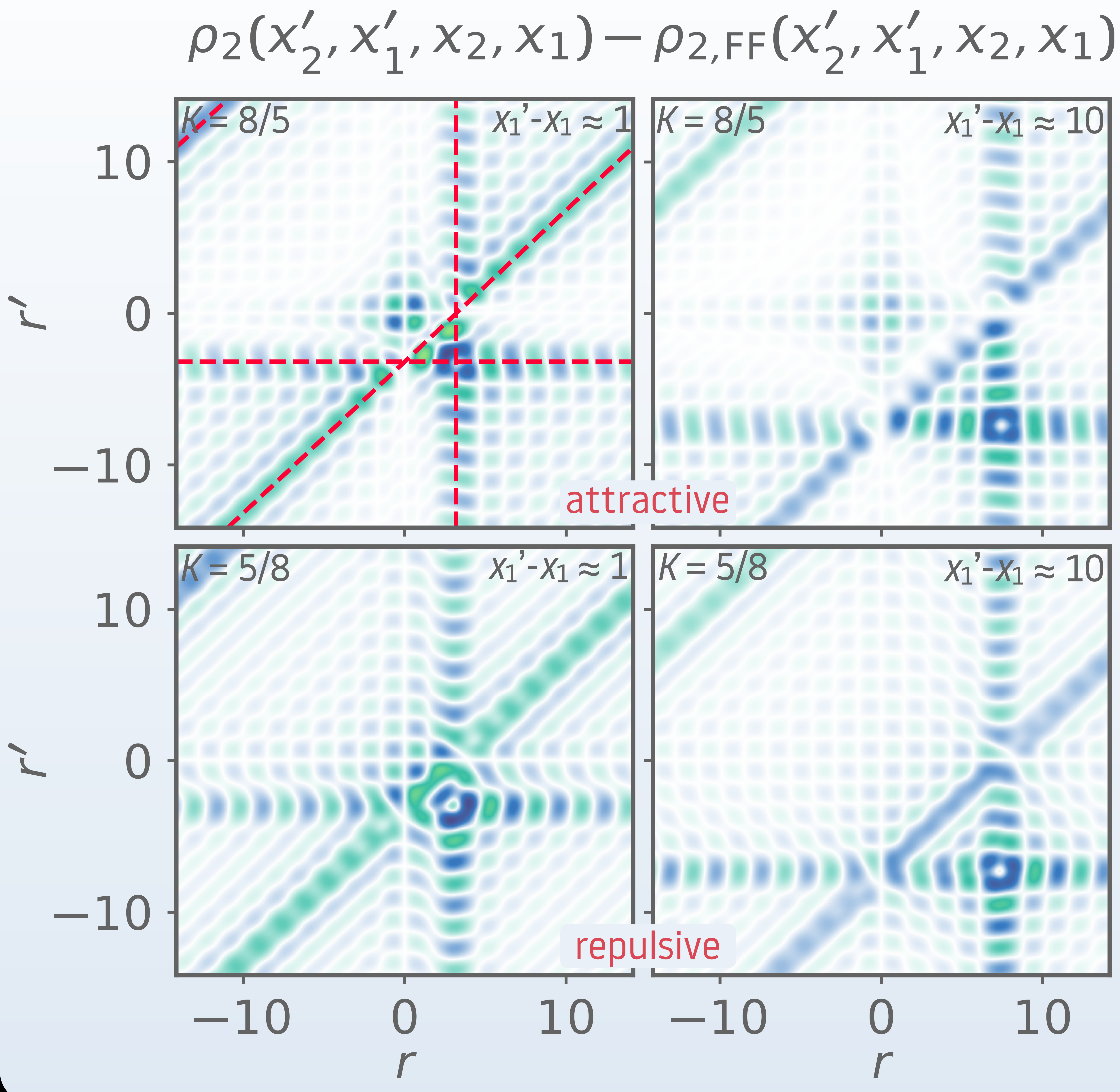
Features: intersections of 4 hyperplanes

$$r_+ = \pm\Delta R$$

$$r_- = \pm\Delta R$$



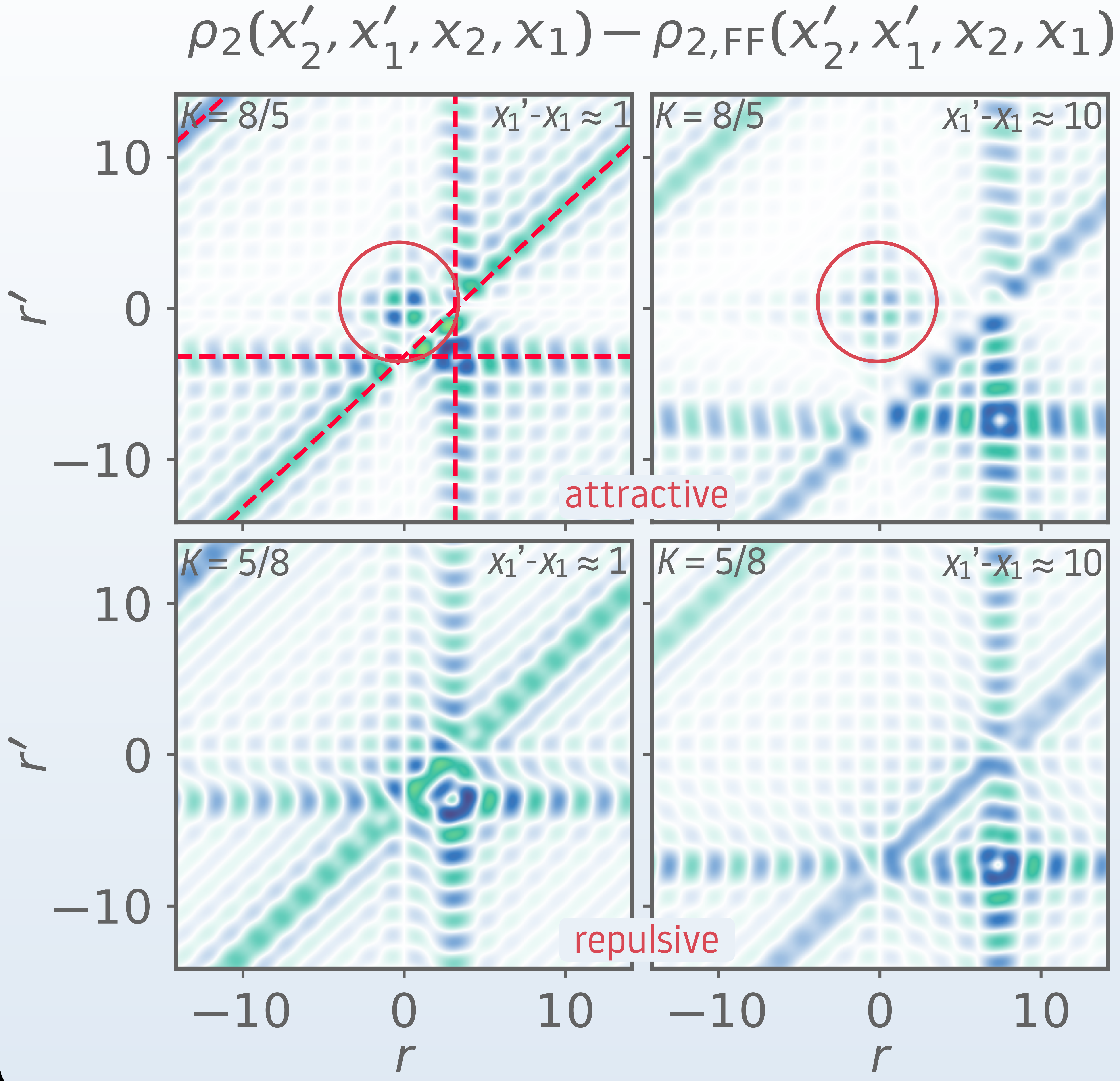
2-RDM: Interaction Effects



Explore: cut // to $x_1' - x_1 = 0$

- Intersections w/ 3 other planes

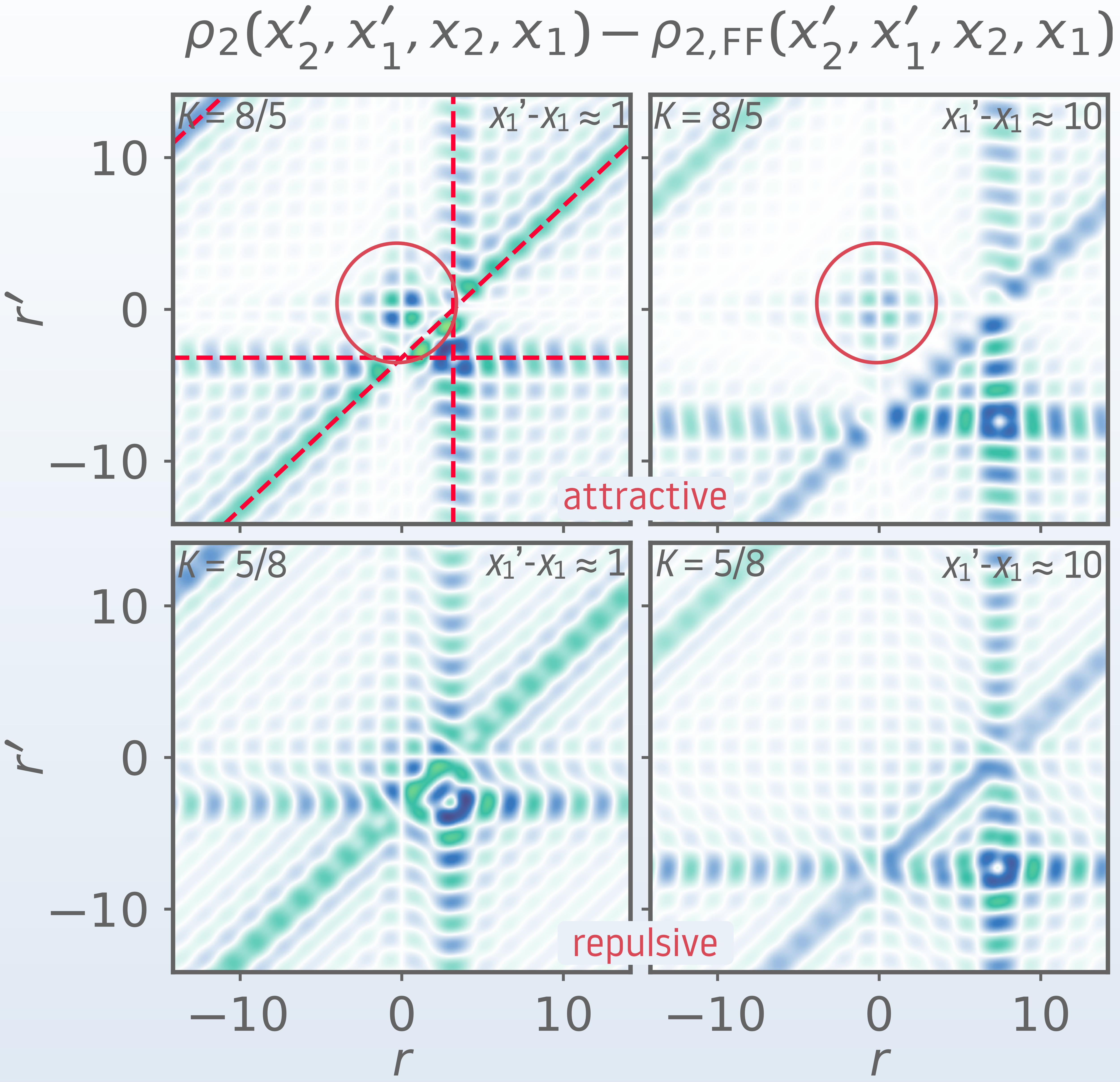
2-RDM: Interaction Effects



Explore: cut // to x_1'-x_1 = 0

- Intersections w/ 3 other planes
- Strong signals of **clustering / pairing**

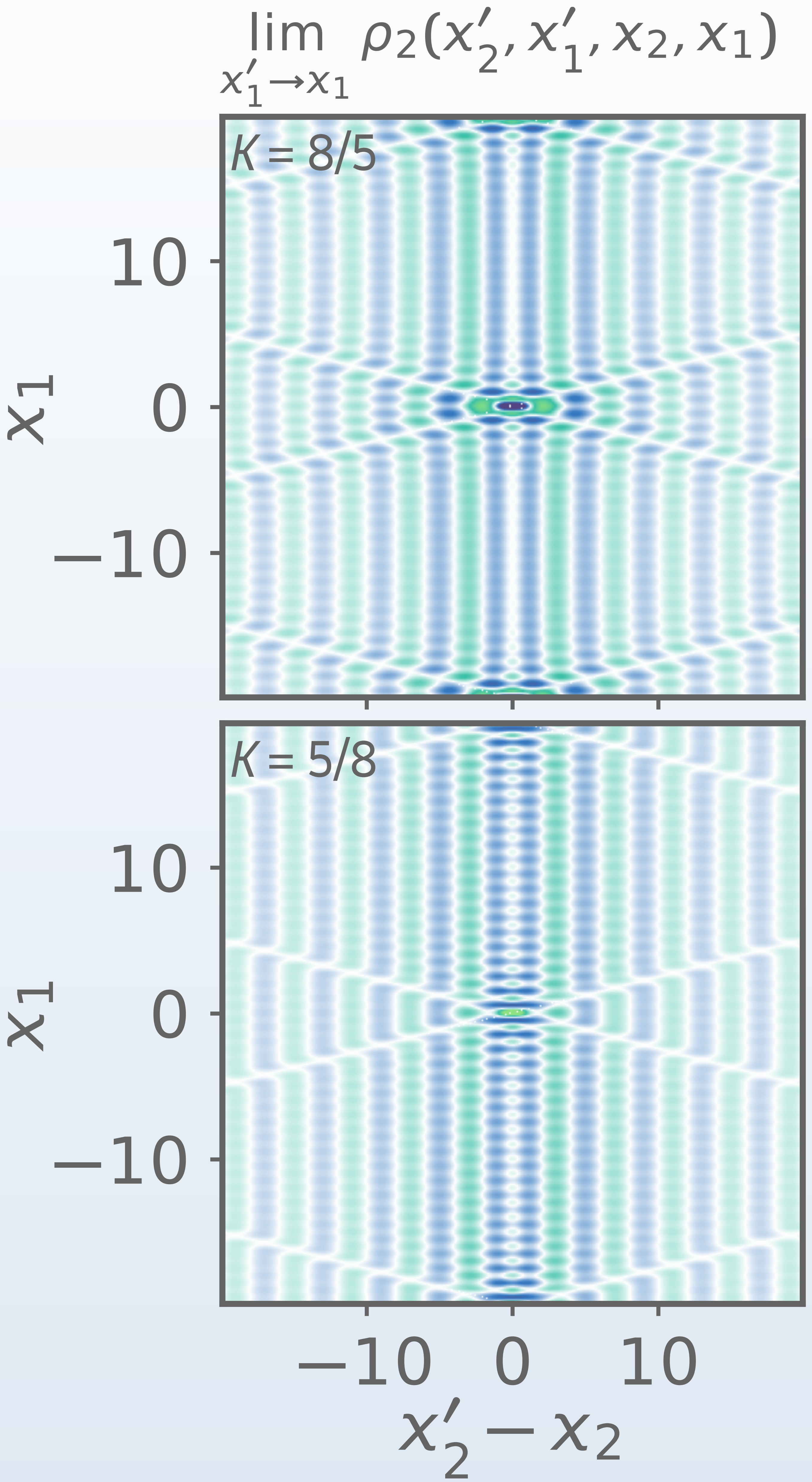
2-RDM: Interaction Effects



Explore: cut // to x_1'-x_1 = 0

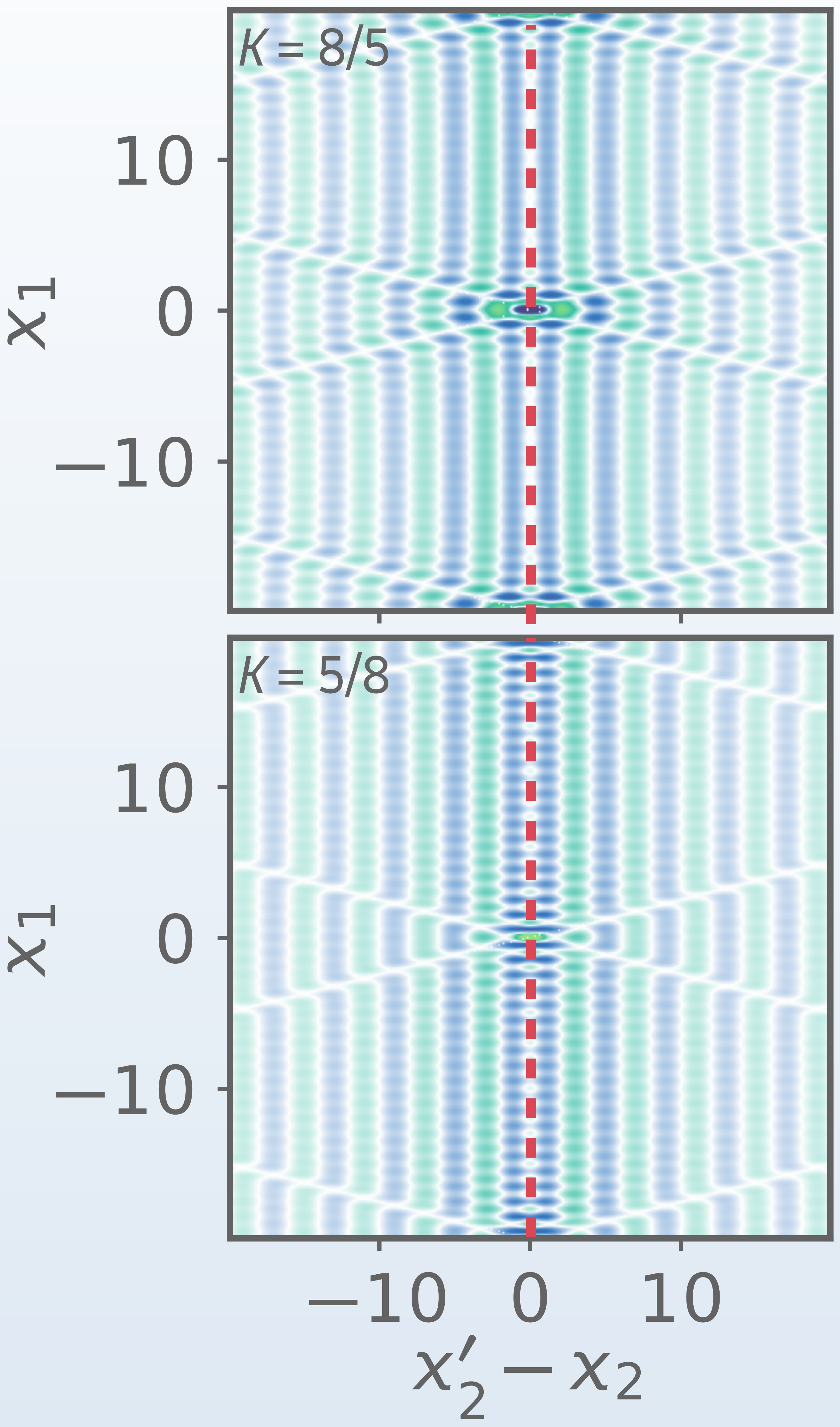
- Intersections w/ 3 other planes
- Strong signals of **clustering / pairing**
- Observe **negative** **copy** of diagonal elements (due to antisymmetrization)

2-RDM: Diagonal Elements



2-RDM: Diagonal Elements

$$\lim_{x_1'\to x_1} \rho_2(x_2', x_1', x_2, x_1)$$

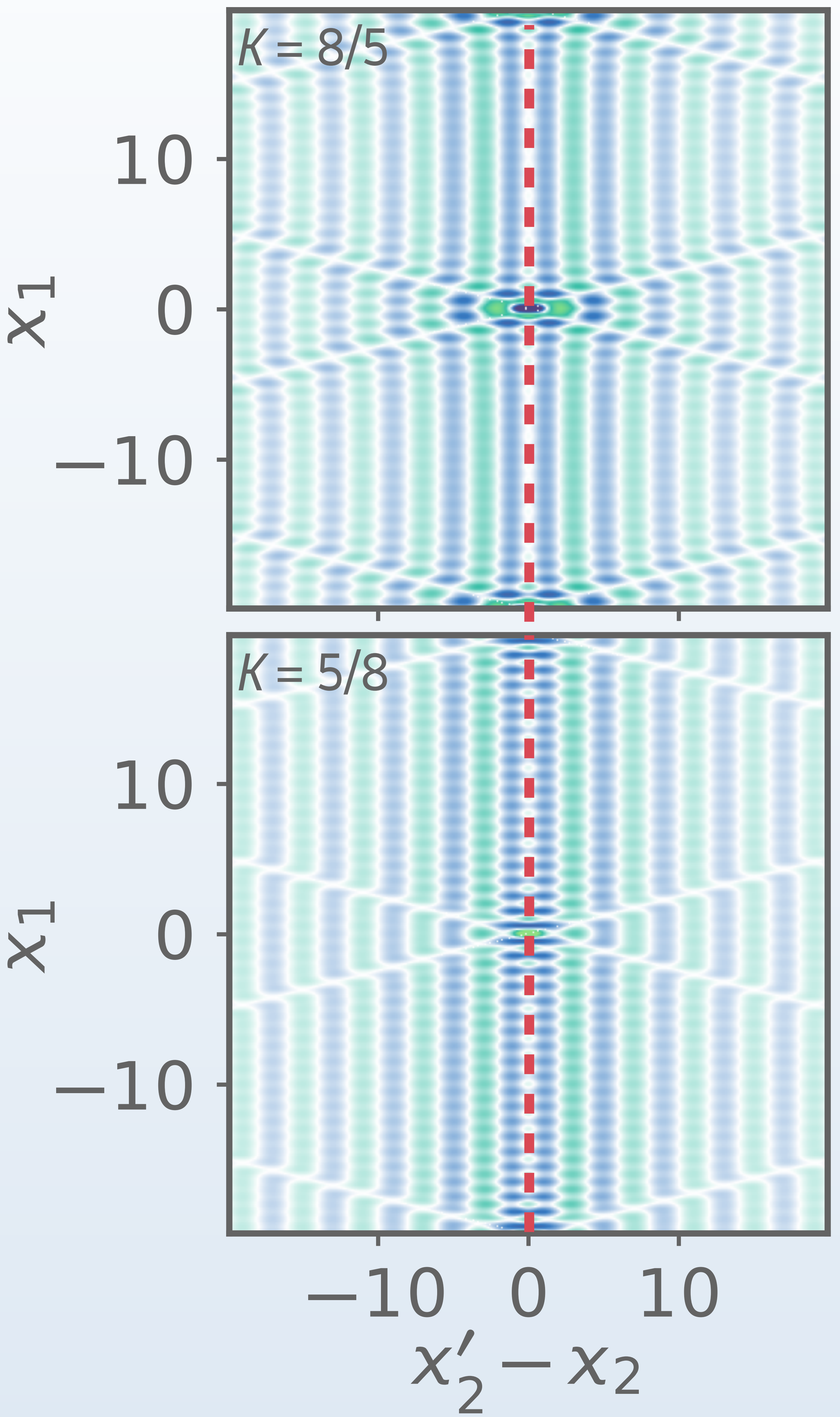


Density-Density correlations

$$\lim_{x_2'\to x_2}\lim_{x_1'\to x_1} \rho_2(x_2', x_1', x_2, x_1) = \rho_0^2 g_2(x_2, x_1)$$
$$= \langle \hat{\varrho}(x_2)\hat{\varrho}(x_1)\rangle - \rho_0\delta(x_1 - x_2)$$

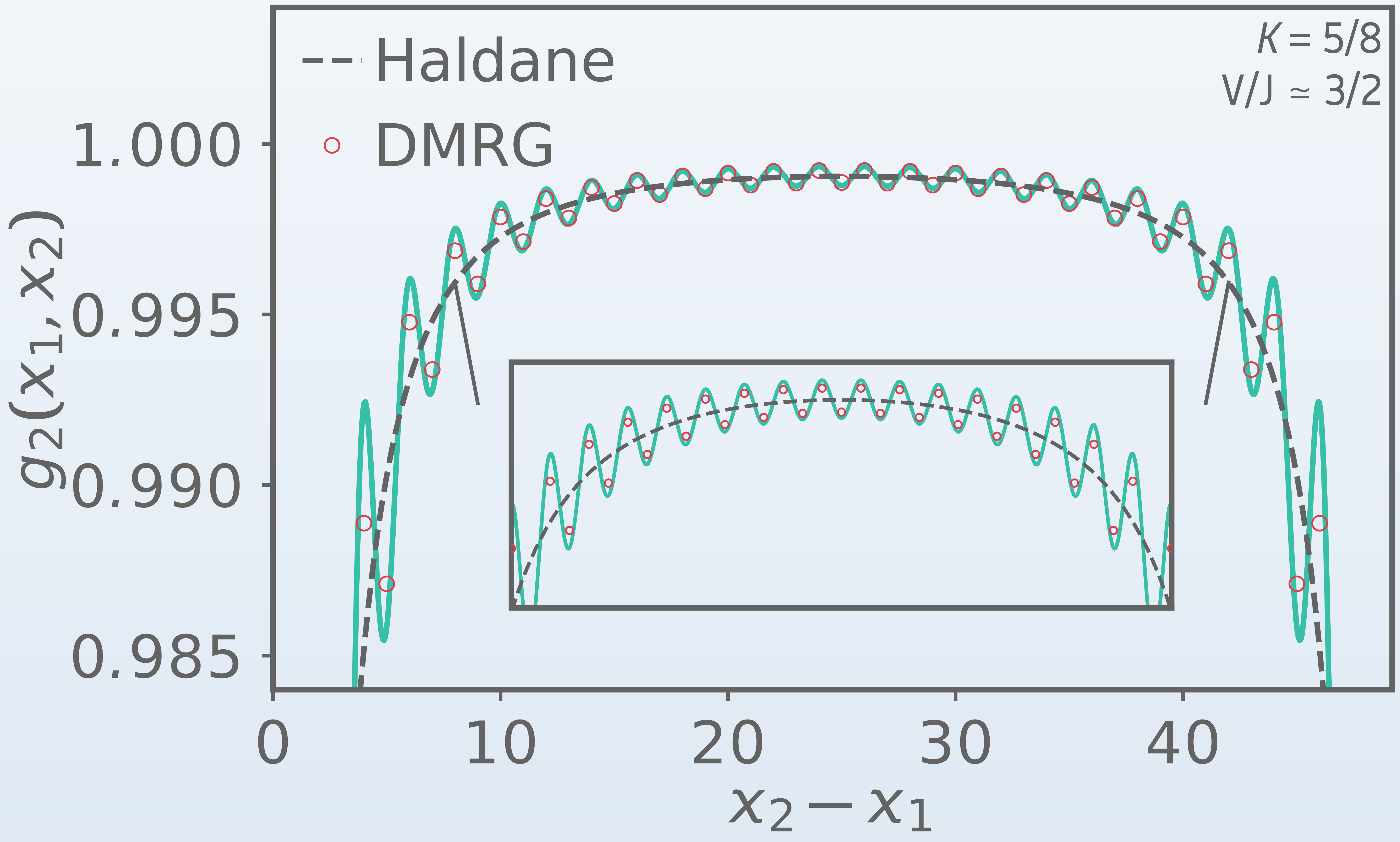
2-RDM: Diagonal Elements

$\lim_{x_1' \to x_1} \rho_2(x_2', x_1', x_2, x_1)$



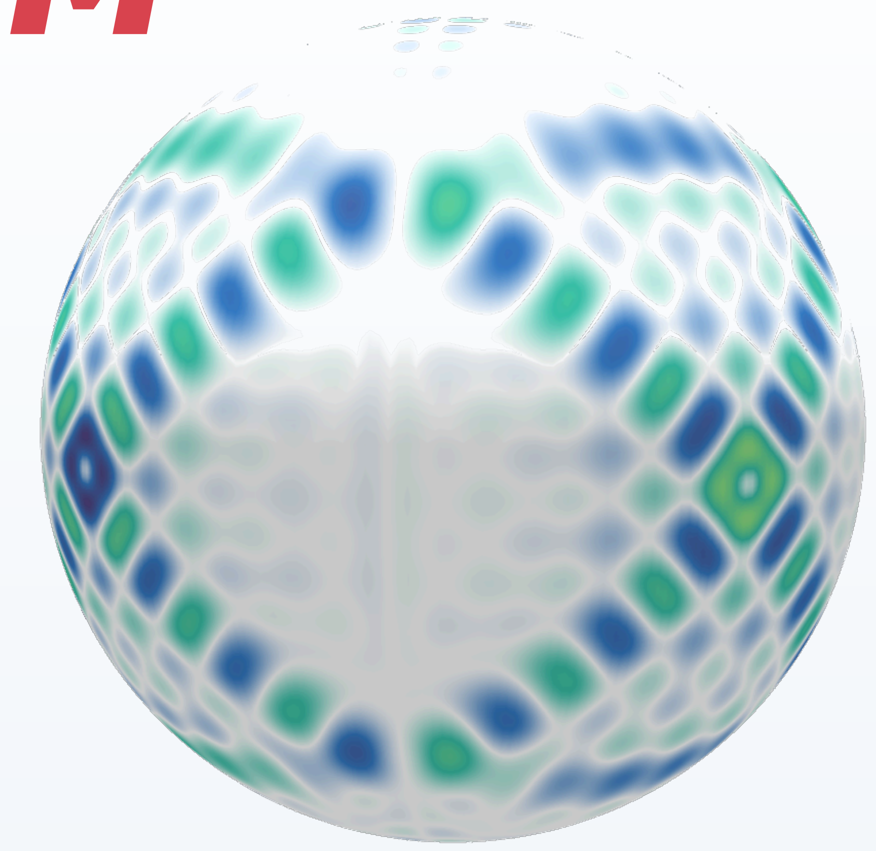
Density-Density correlations

$$\lim_{x_2' \to x_2} \lim_{x_1' \to x_1} \rho_2(x_2', x_1', x_2, x_1) = \rho_0^2 g_2(x_2, x_1)$$
$$= \langle \hat{\varrho}(x_2) \hat{\varrho}(x_1) \rangle - \rho_0 \delta(x_1 - x_2)$$



A Finite Size LL Expression for the 2-RDM

A theoretical playground for probing interaction effects in the quantum liquid regime of a 1D interacting system (e.g. deviation from Wick's theorem, pairing signatures, …)



Open Questions & Future Prospects

- Particle entanglement: DMRG results hint at the possibility of universal scaling with N.
- Use ρ_2 to probe the Carlen-Lieb-Reuvers conjecture.
- …