Quantum many-body scars in dual-unitary circuits

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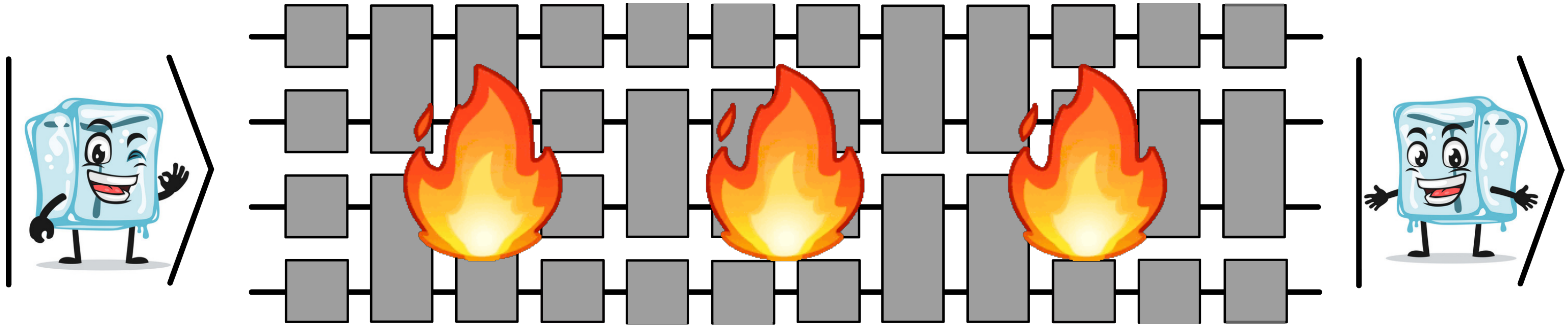


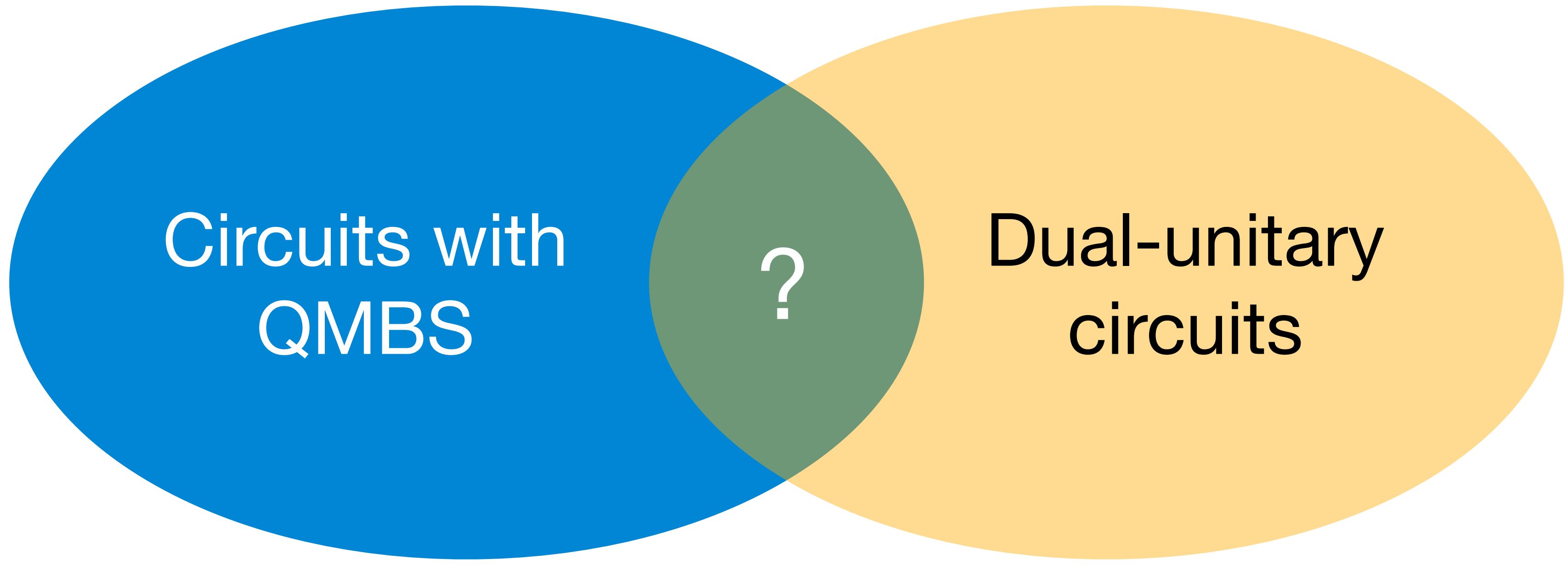
Shane Dooley — Dublin Institute for Advanced Studies / Trinity College Dublin

(with Leonard Logaric, Silvia Pappalardi and John Goold)

26 Sep 2024 — RPMBT22, Tsukuba

Outline

1. Background/motivation: Thermalisation, chaos, ETH, QMBS
2. Dual-unitary (DU) circuits: exact results — “maximally chaotic”, fast thermalisers
3. Embedding QMBS in DU circuits: avoiding thermalisation for certain initial states

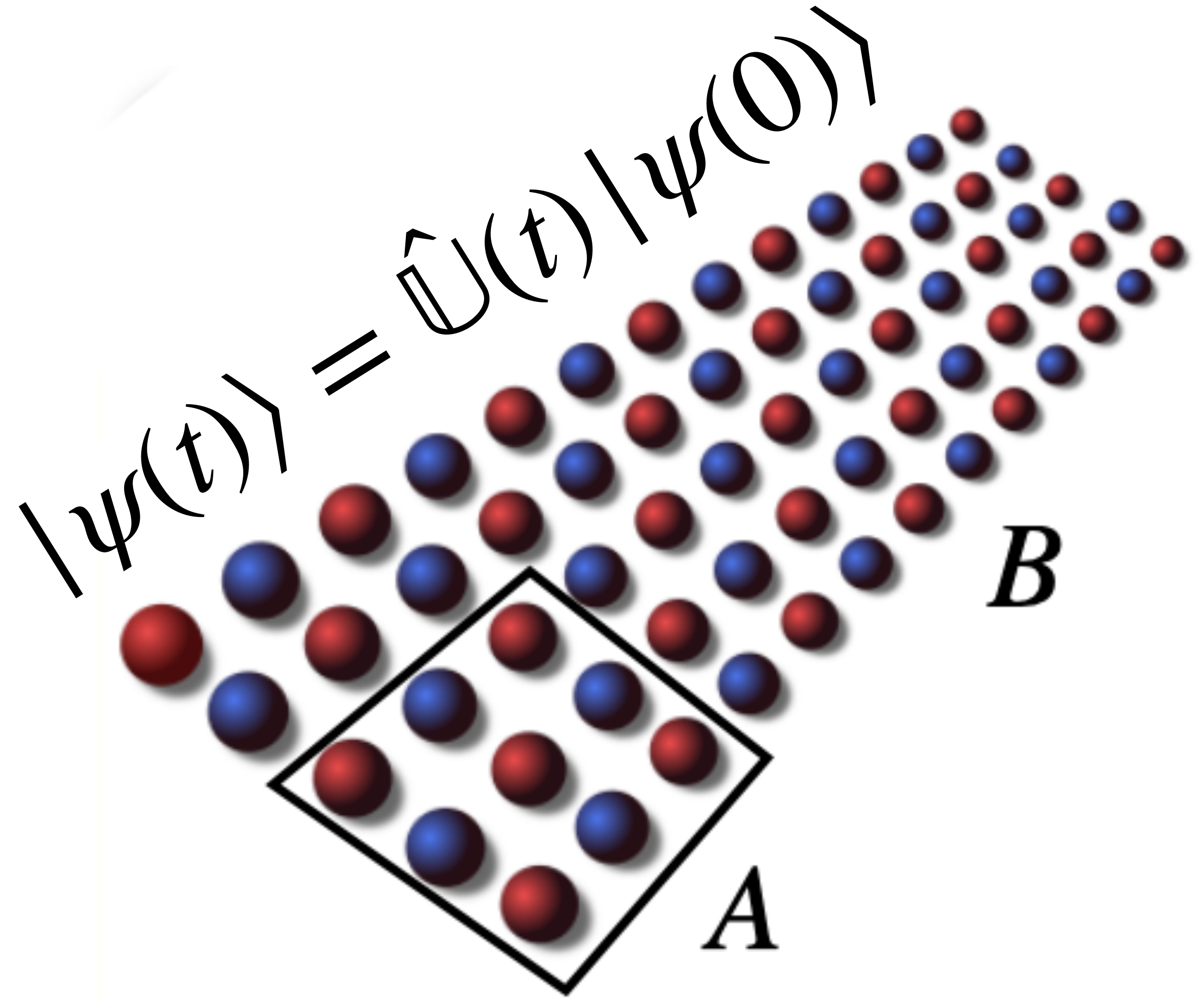


1. Background: Thermalisation/chaos/ETH/QMBS

Thermalisation of closed many-body systems

- Thermal equilibrium (according to quantum stat. mech) ***versus*** unitary evolution

  - $\hat{\mathbb{U}}(t) = \exp\left[-\frac{i}{\hbar}\hat{\mathbb{H}}\right] \implies \hat{\rho}_{\text{thermal}} = \frac{1}{Z}\exp[-\beta\hat{\mathbb{H}}]$
  - Floquet $\hat{\mathbb{H}}(t) = \hat{\mathbb{H}}(t+T) \implies \hat{\rho}_{\text{thermal}} = \frac{1}{D}\hat{\mathbb{I}}$



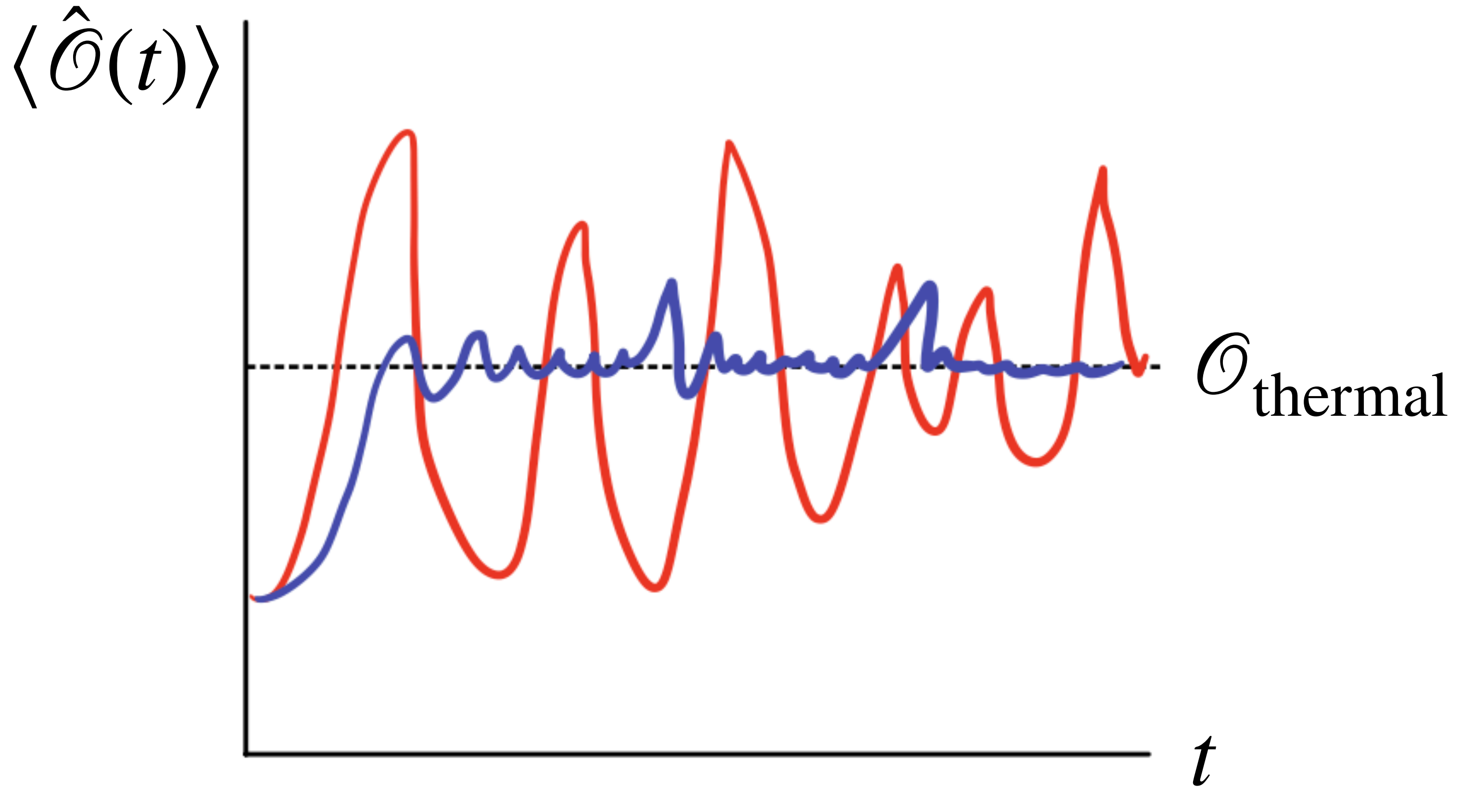
- Evolving pure state $|\psi(t)\rangle$ will never approach a highly mixed $\hat{\rho}_{\text{thermal}}$

- Instead, consider **thermalisation of observables**: $\langle\mathcal{O}(t)\rangle = \langle\psi(t)|\hat{\mathcal{O}}|\psi(t)\rangle$ ***versus*** $\mathcal{O}_{\text{thermal}} = \text{Tr}[\mathcal{O}\rho_{\text{thermal}}]$

  - Weak thermalisation (ergodicity):

$$\overline{\langle\mathcal{O}\rangle} = \lim_{T\to\infty}\frac{1}{T}\int_0^T dt\,\langle\mathcal{O}(t)\rangle \approx \mathcal{O}_{\text{thermal}}$$

  - Strong thermalisation: $\langle\mathcal{O}(t)\rangle \approx \mathcal{O}_{\text{thermal}}$ for most times t



- System is thermalised if all "realistic" (few-body) observables are thermalised

Chaos and thermalisation

- Not all many-body systems thermalise, e.g., **integrable** and **many-body localised** (MBL) systems:

  An extensive number of local conserved quantities prevent thermalisation: $[\hat{\mathbb{U}}, \hat{\mathcal{O}}] = 0 \implies \langle \hat{\mathcal{O}}(t) \rangle = \langle \hat{\mathcal{O}}(0) \rangle$

- Many numerical studies show **thermalisation for chaotic** many-body systems

- Classical chaos: sensitivity to initial conditions $|\delta r(t)| \sim e^{\lambda t} |\delta r(0)|$

  Doesn't translate directly to quantum systems:

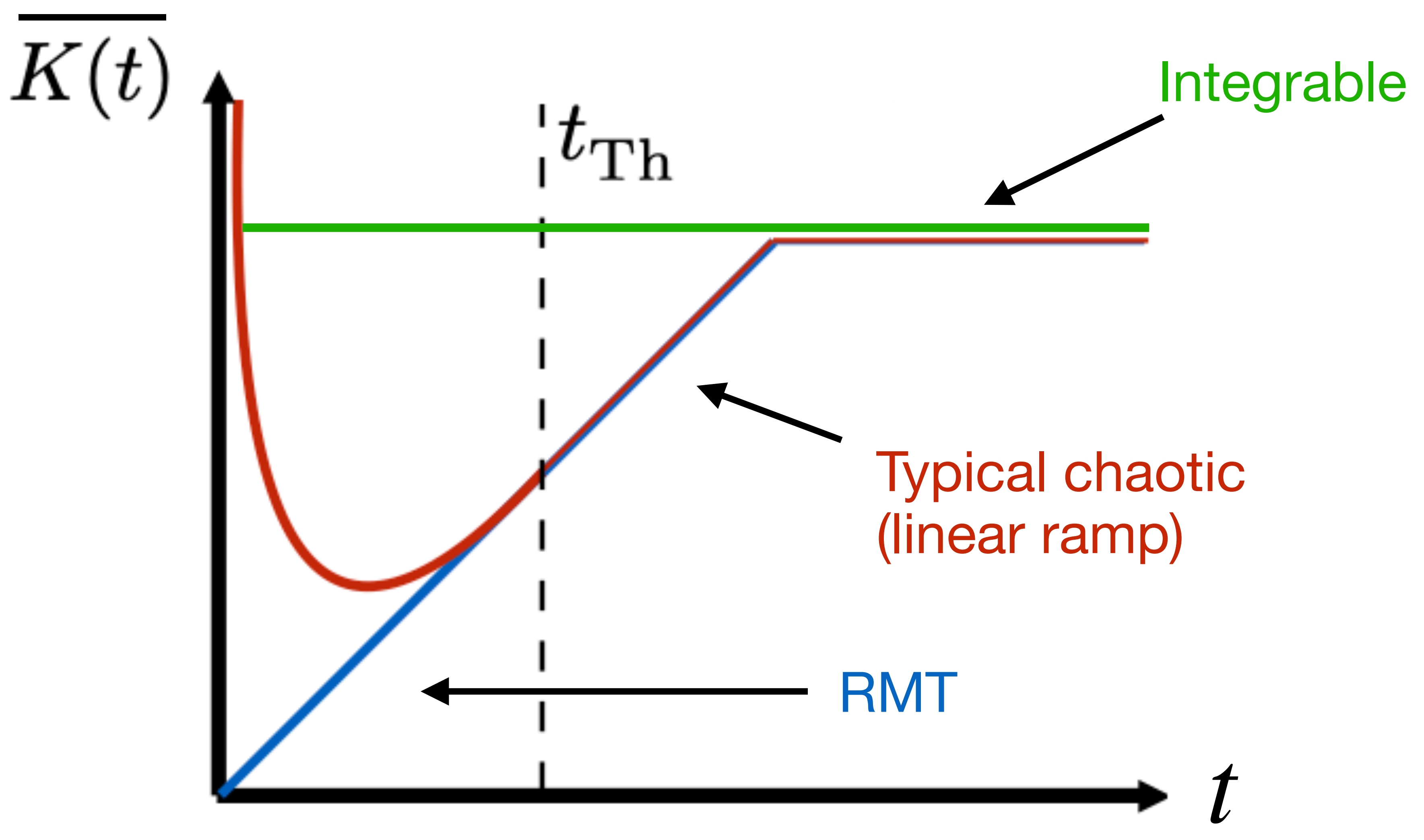
$$|\langle \psi_1(t) | \psi_2(t) \rangle| = |\langle \psi_1(0) | \psi_2(0) \rangle|$$

- What is quantum chaos?

  Random-matrix-like correlations in (quasi-) energy spectrum

  [Bohigas-Giannoni-Schmit (BGS) conjecture (1984)]

e.g., **spectral form factor** $K(t) = |\mathrm{Tr}[\mathbb{U}(t)]|^2$



Qu. chaos not sufficient for thermalisation: Harvard/MIT experiment

Probing many-body dynamics on a 51-atom quantum simulator

Hannes Bernien, Sylvain Schwartz, Alexander Keesling, Harry Levine, Ahmed Omran, Hannes Pichler, Soonwon Choi, Alexander S. Zibrov, Manuel Endres, Markus Greiner, Vladan Vuletić & Mikhail D. Lukin

Nature **551**, 579–584 (30 November 2017) | Download Citation

nature physics
ARTICLES
https://doi.org/10.1038/s41567-018-0137-5

Weak ergodicity breaking from quantum many-body scars

C. J. Turner[1], A. A. Michailidis[1,2], D. A. Abanin[3], M. Serbyn[2] and Z. Papić[1*]

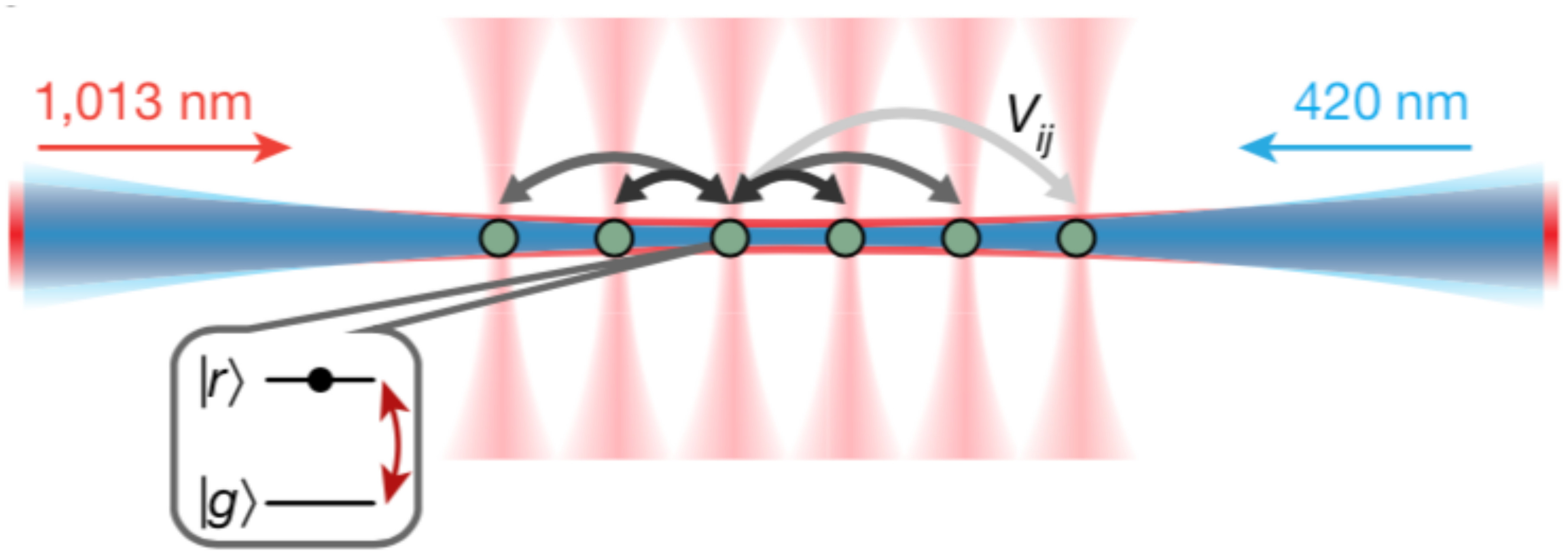
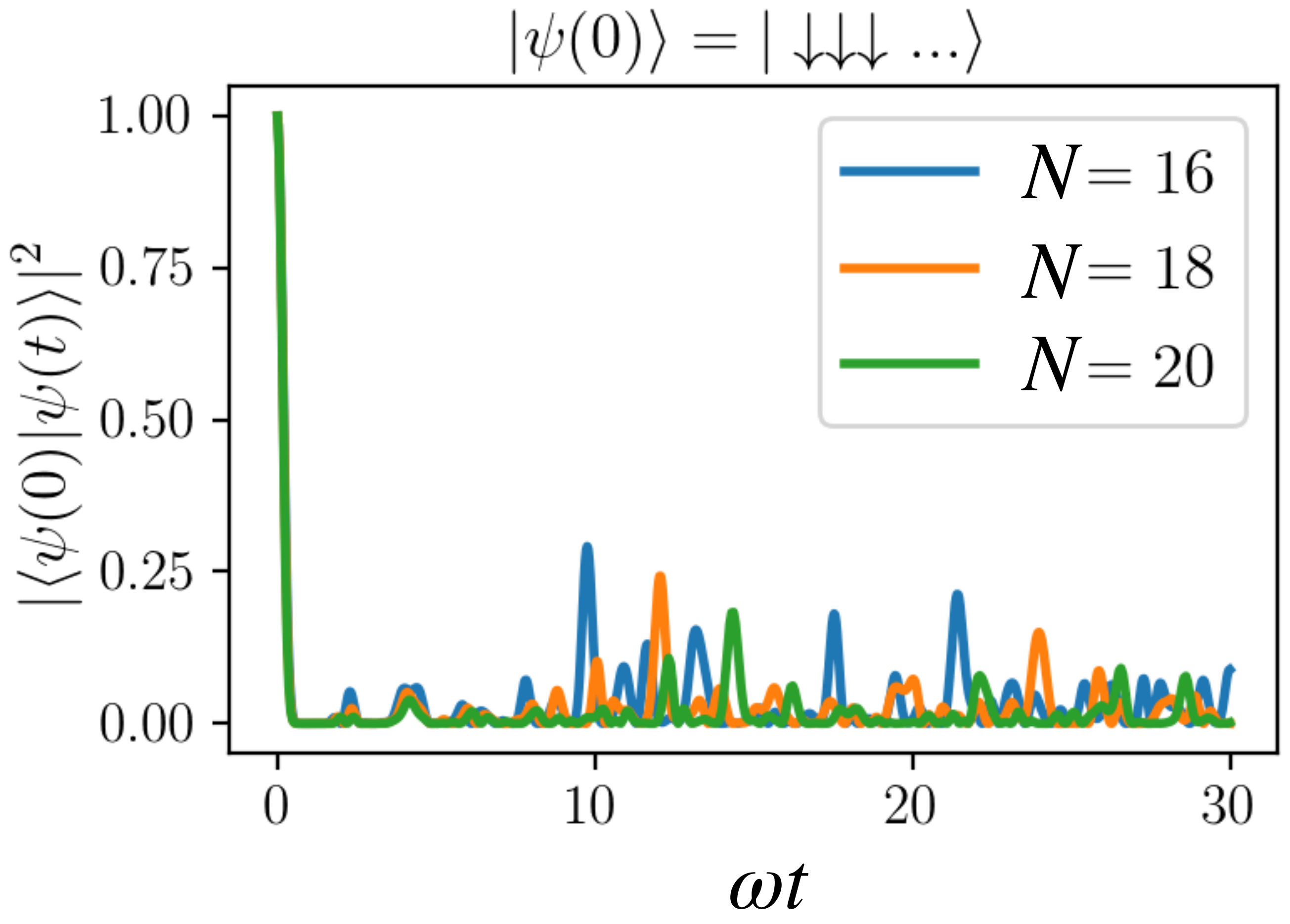
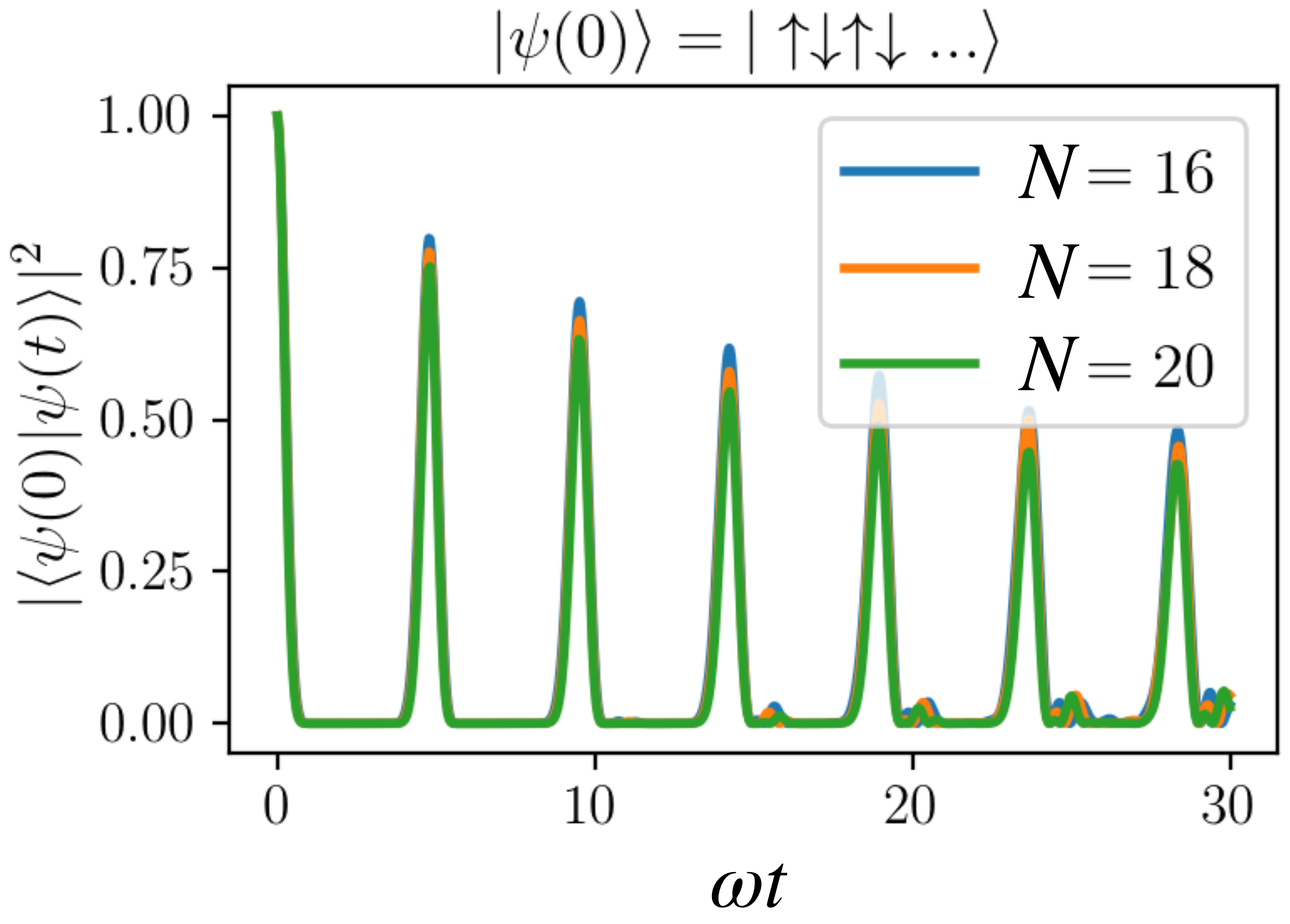
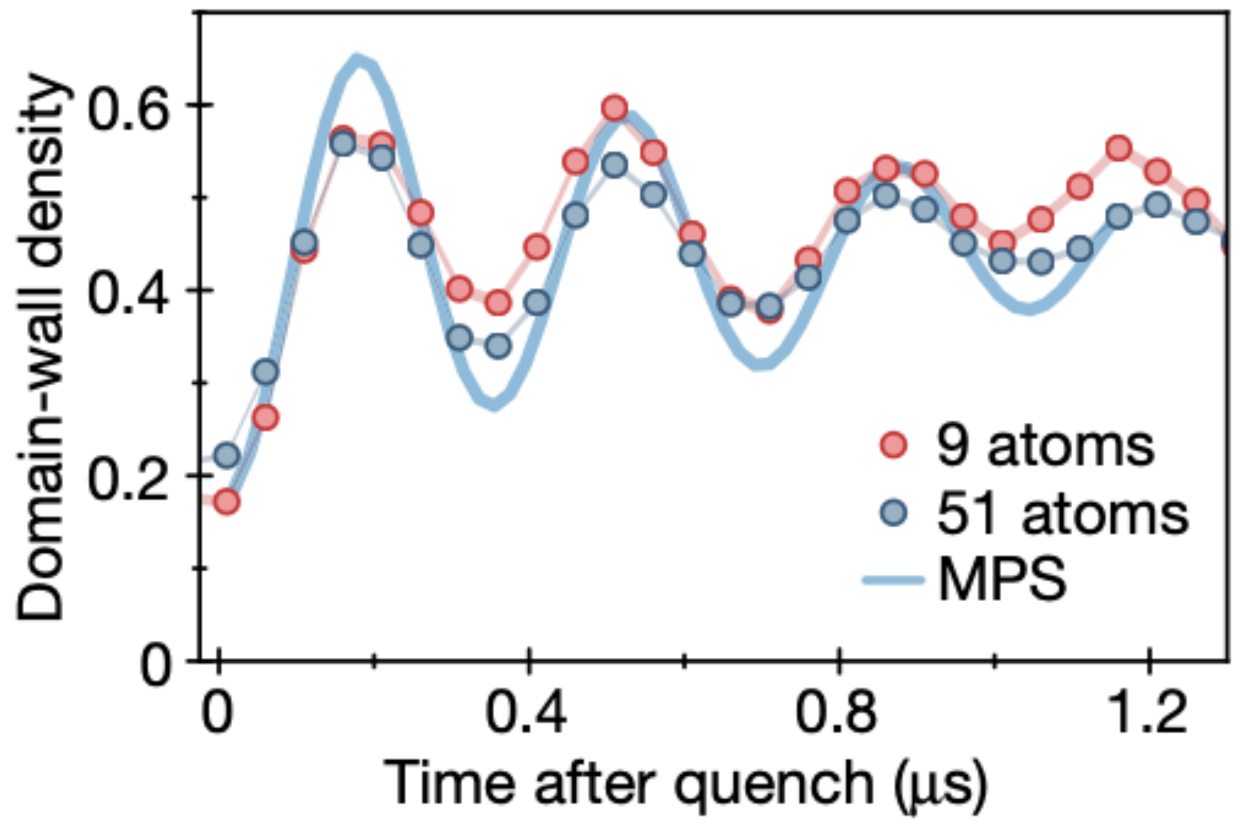


Figure 1 | Experimental platform. a, Individual ^{87}Rb atoms (green) are trapped using optical tweezers (vertical red beams) and arranged into defect-free arrays. Coherent interactions V_{ij} between the atoms (arrows) are enabled by exciting them (horizontal blue and red beams) to a Rydberg state with strength Ω and detuning Δ (inset).







Eigenstate thermalisation hypothesis (ETH)

- A more precise criteria for thermalisation: the **eigenstate thermalisation hypothesis**.

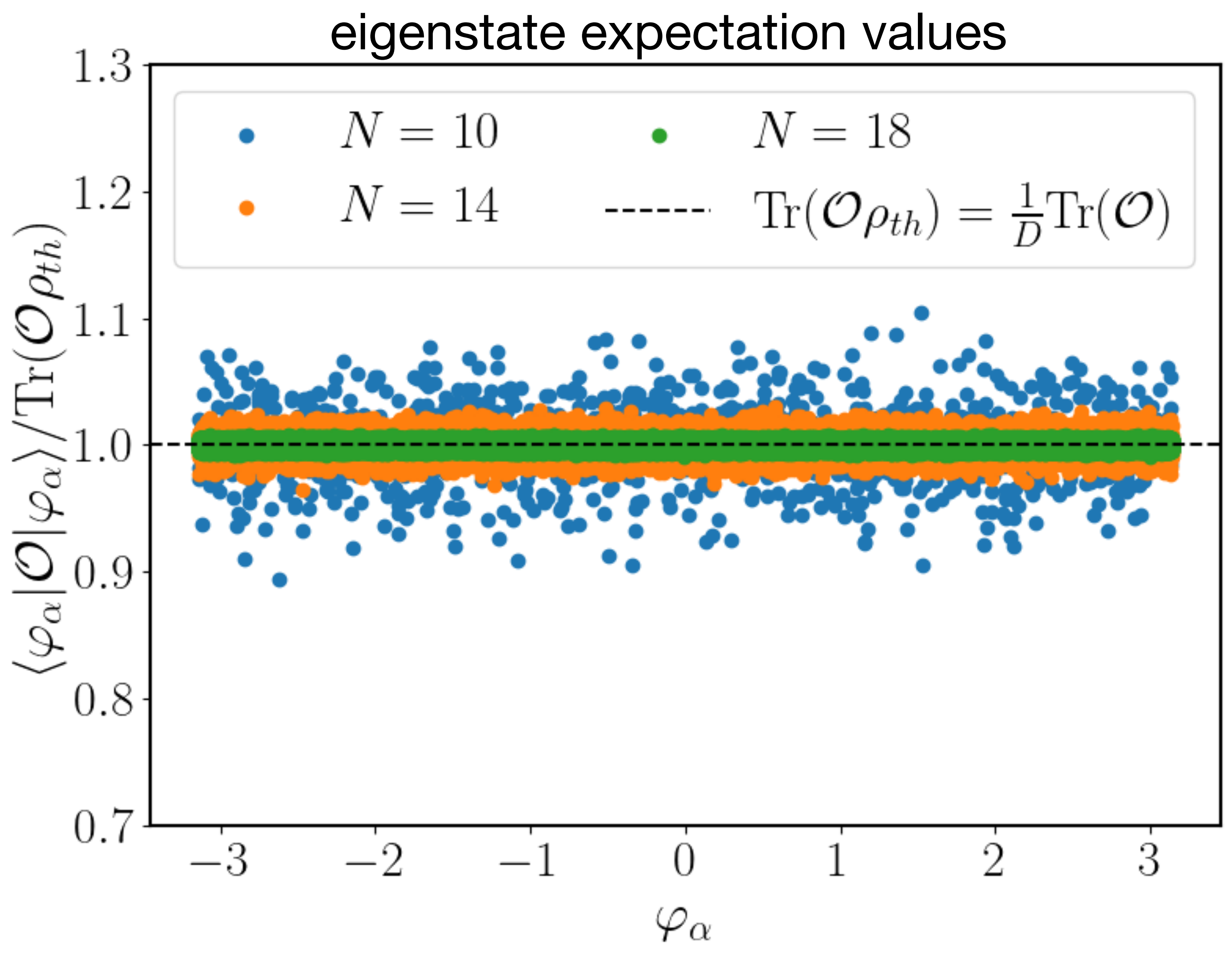
For example, for a Floquet unitary $\hat{\mathbb{U}} = \mathcal{T}\exp\left[-\frac{i}{\hbar}\int_0^T \hat{\mathbb{H}}(\tau)d\tau\right]$ with $\hat{\mathbb{U}}\,|\,\varphi_\alpha\rangle = e^{i\varphi_\alpha}\,|\,\varphi_\alpha\rangle$:

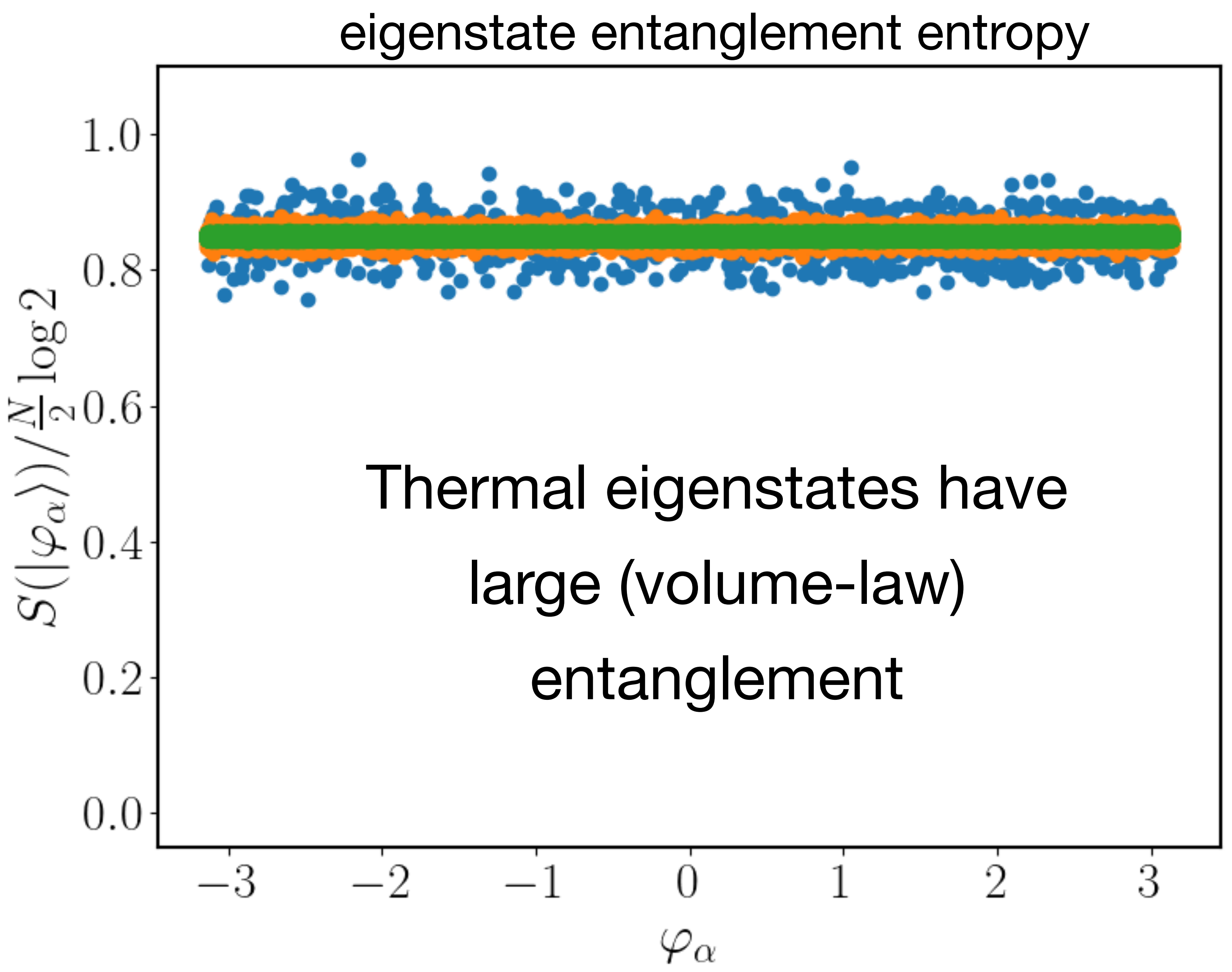
ETH ansatz:

$$\langle\varphi_\alpha\,|\,\hat{\mathcal{O}}\,|\,\varphi_{\alpha'}\rangle = \frac{1}{D}\mathrm{Tr}(\hat{\mathcal{O}})\delta_{\alpha,\alpha'} + \frac{1}{\sqrt{D}}f_{\mathcal{O}}(\varphi_\alpha - \varphi_{\alpha'})R_{\alpha,\alpha'}$$

$$\overset{(\alpha=\alpha')}{\Longrightarrow} \quad \langle\varphi_\alpha\,|\,\hat{\mathcal{O}}\,|\,\varphi_\alpha\rangle = \overbrace{\frac{1}{D}\mathrm{Tr}(\hat{\mathcal{O}})}^{\mathcal{O}_{\text{thermal}} = \mathrm{Tr}[\hat{\mathcal{O}}\hat{\rho}_{\text{thermal}}]} + \frac{1}{\sqrt{D}}R_{\alpha,\alpha'}$$

- All eigenstates are thermal (i.e., obey ETH ansatz) w.r.t. $\hat{\mathcal{O}} \Longrightarrow$ strong thermalisation of $\langle\hat{\mathcal{O}}\rangle$!

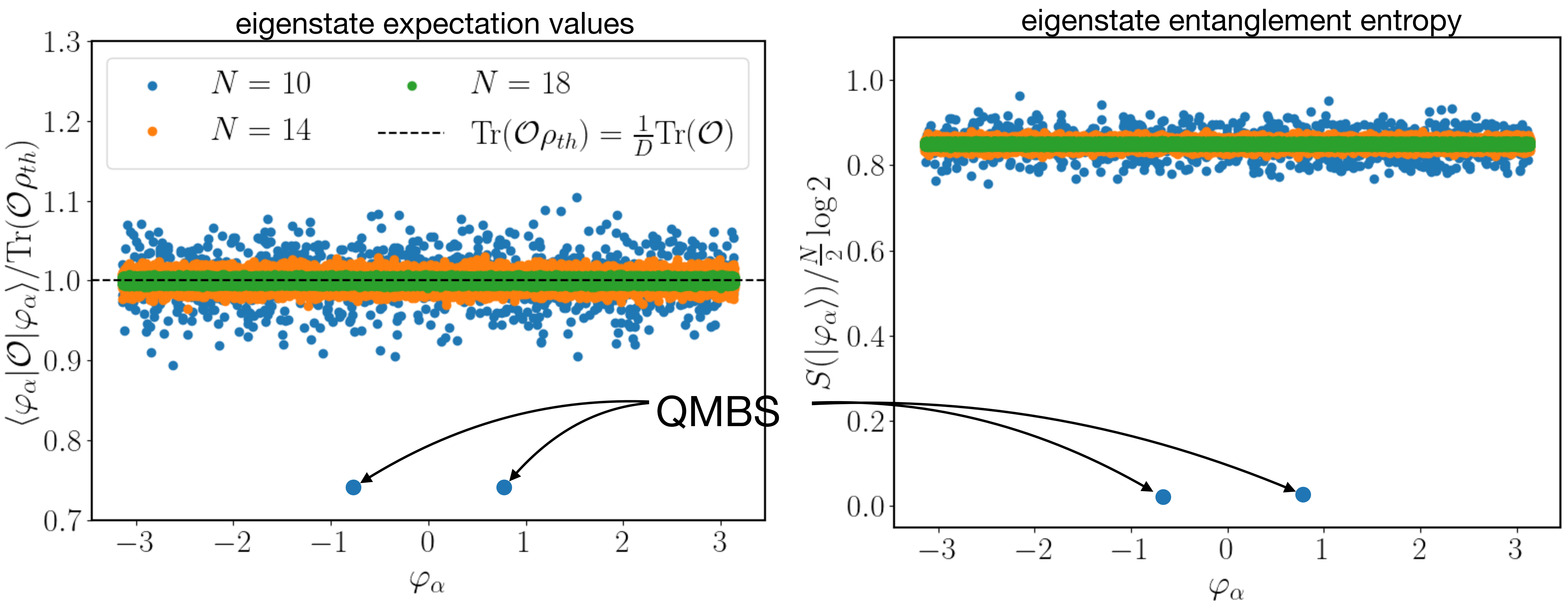




[D'Alessio *et al.*, Advances in Physics, 2016 (Review)]

Violations of the ETH

- Thermalisation can be avoided if there are "non-thermal" (i.e., ETH-violating) eigenstates
- Two possibilities:
  - **Strong** ETH-violation: significant fraction of eigenstates are non-thermal (e.g., integrable systems, MBL)
  - **Weak** ETH-violation: non-thermal eigenstates are rare — quantum many-body scars (QMBS)
    Can prevent thermalisation if QMBS have large overlap with $|\psi(0)\rangle$



[Shiraishi & Mori, Phys. Rev. Lett. 119, 030601, 2017]

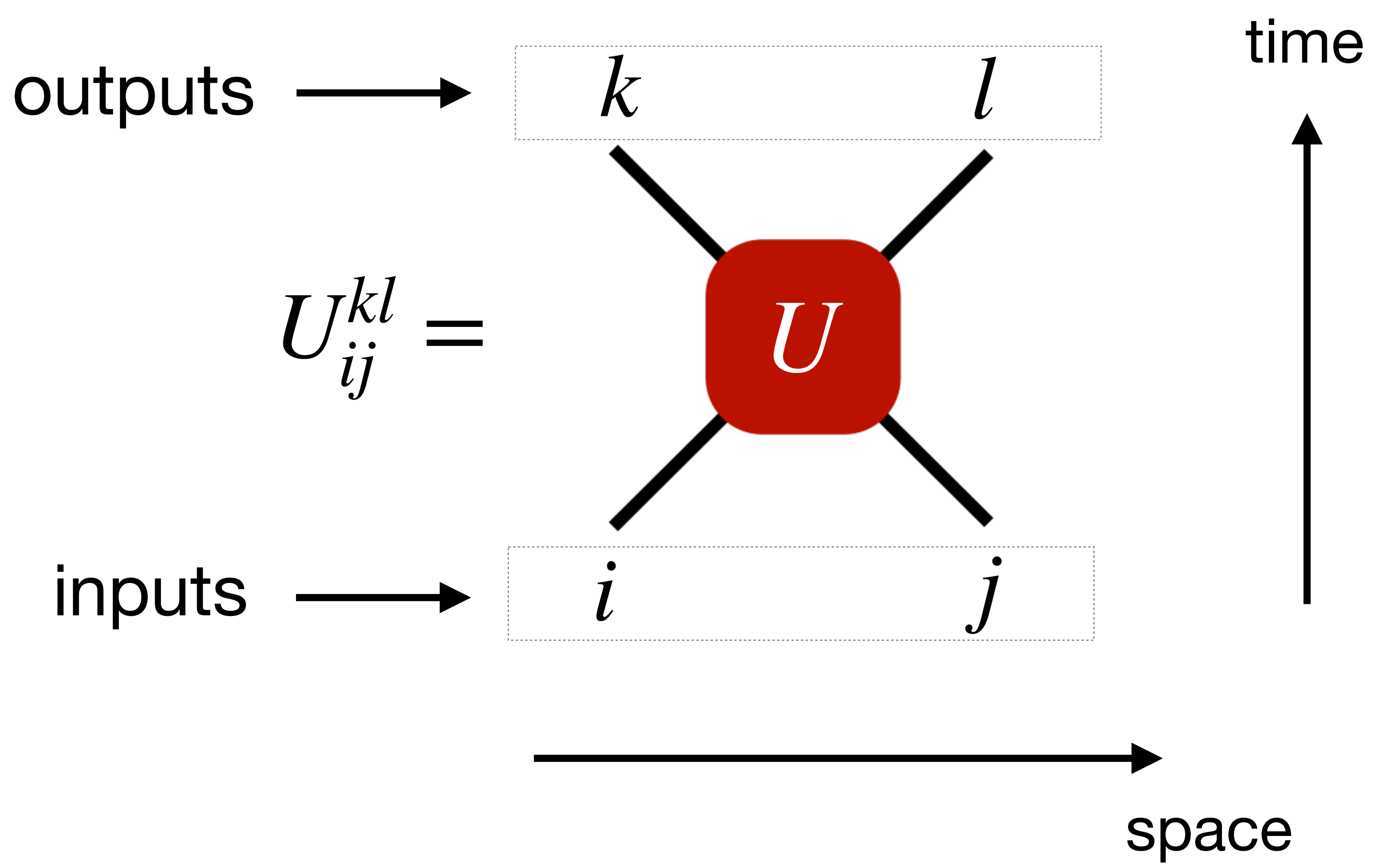
2. Dual-unitary circuits

Dual unitarity

Consider a bipartite unitary operator U:

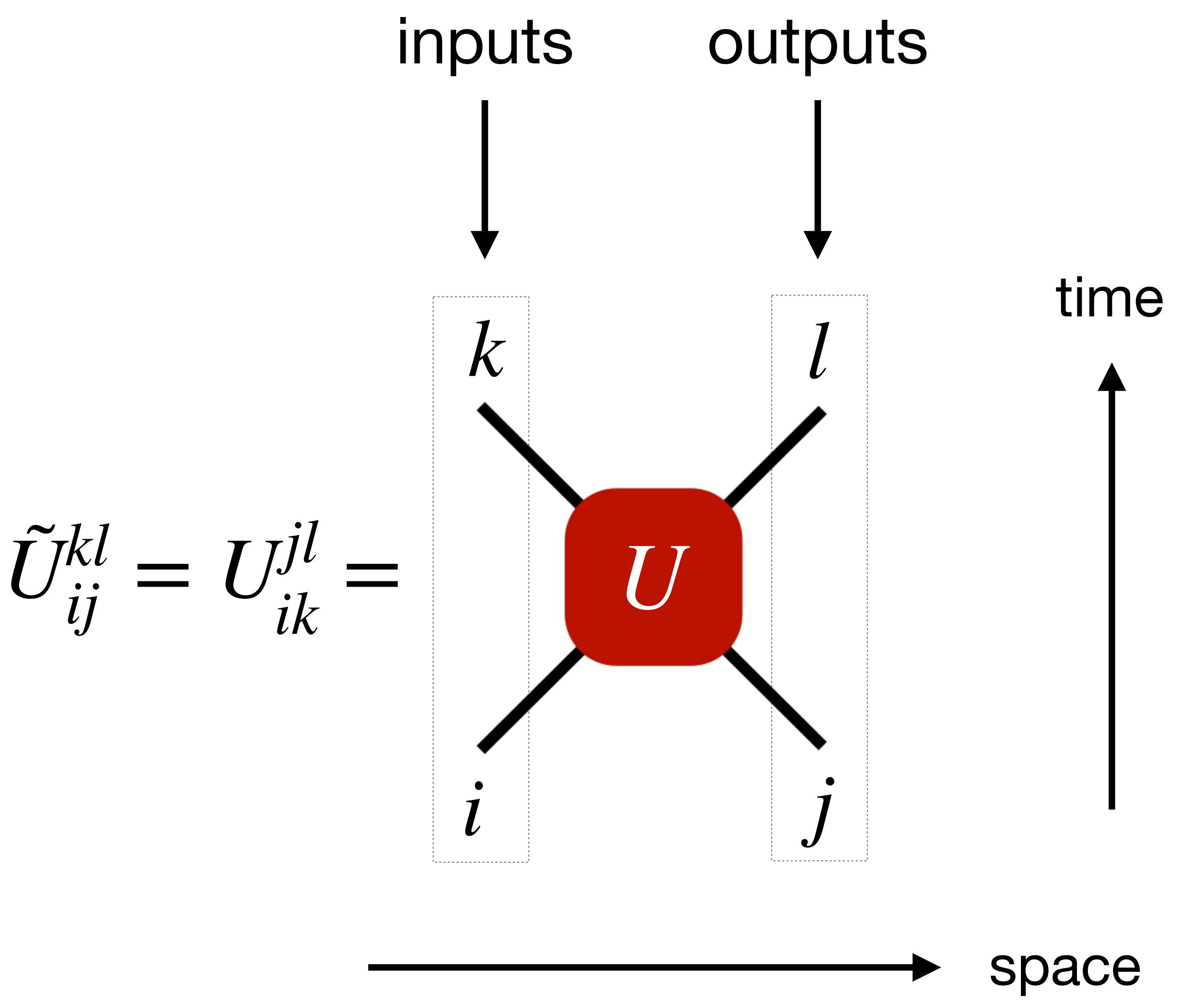
$$U : \mathscr{H} \otimes \mathscr{H} \to \mathscr{H} \otimes \mathscr{H}, \qquad \dim \mathscr{H} = d$$

$$U = \sum_{i,j,k,l} U_{ij}^{kl} \, |k\rangle\langle i| \otimes |l\rangle\langle j|, \qquad U^\dagger U = U U^\dagger = \mathbb{I}$$



Define a new "dual" operator $\tilde{U}$ by a reordering of input/output indices:

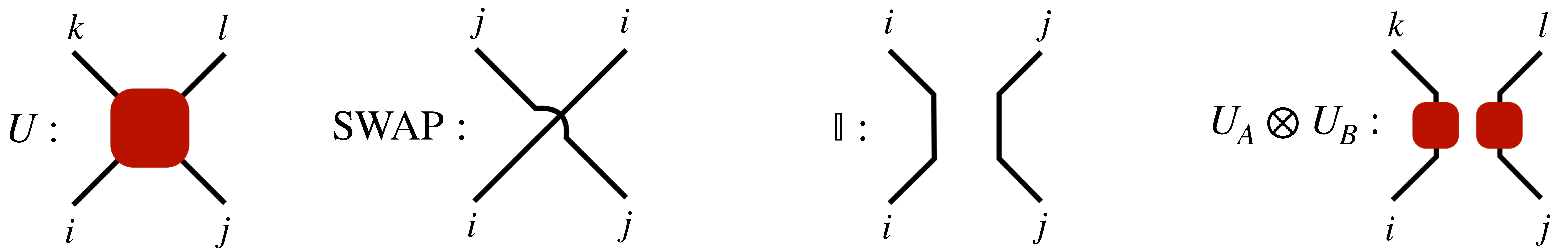
$$\tilde{U} = \sum_{i,j,k,l} U_{ij}^{kl} \, |j\rangle\langle i| \otimes |l\rangle\langle k| \quad = \sum_{i,j,k,l} U_{ik}^{jl} \, |k\rangle\langle i| \otimes |l\rangle\langle j|$$



If $\tilde{U}$ is unitary then U is called dual unitary.

Examples / parameterisation?

- Examples: dual unitary ✓ not dual unitary ✗ not dual unitary ✗



- Fully classified for $d = 2$: $U^{DU} = e^{i\phi}(u_+ \otimes u_-)U_{XXZ}[J](v_+ \otimes v_-)$ (14 free parameters)

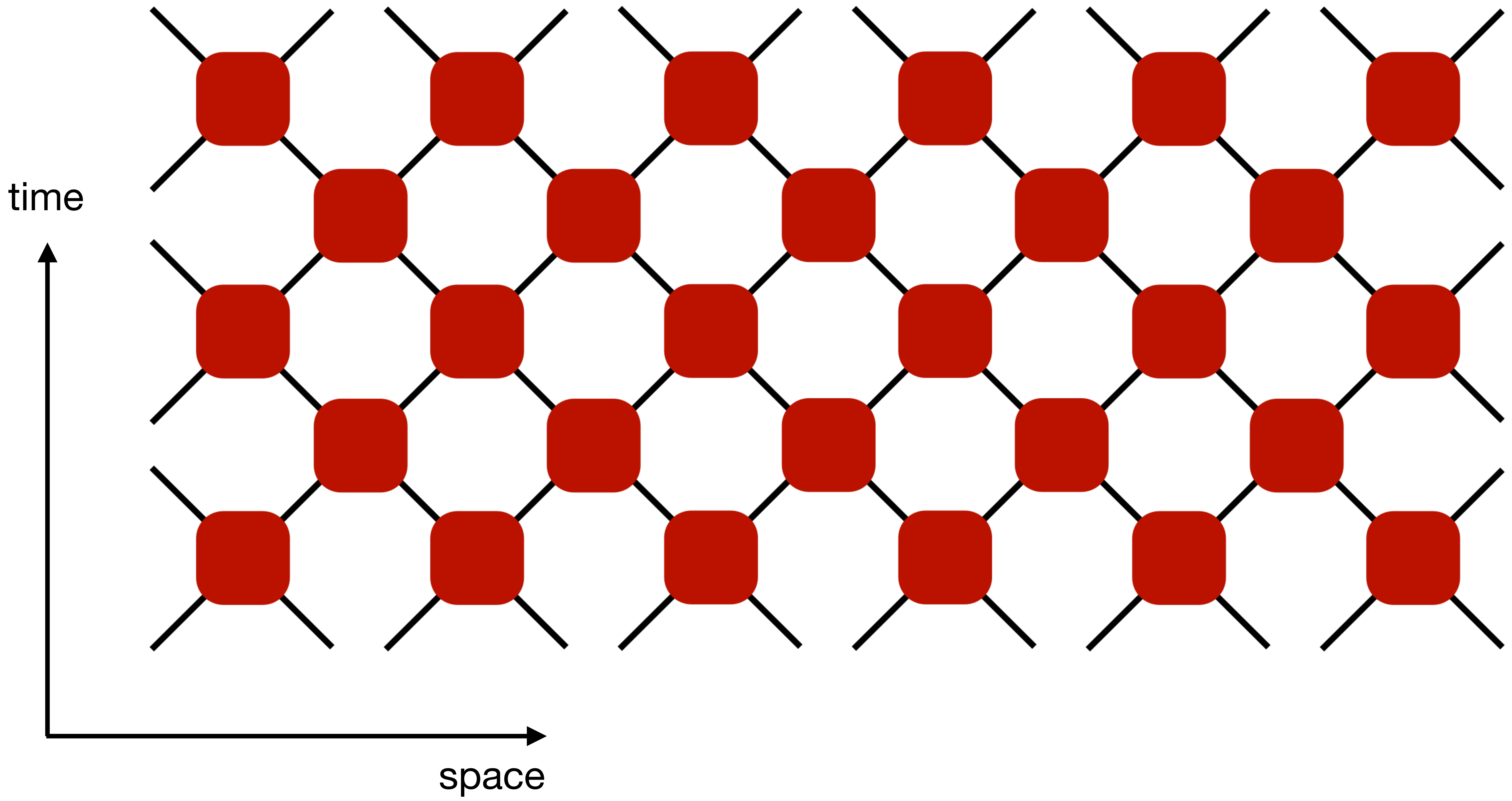
$$U_{XXZ}[J] = \exp\left\{ -i\left(\frac{\pi}{4}\sigma^x \otimes \sigma^x + \frac{\pi}{4}\sigma^y \otimes \sigma^y + J\sigma^z \otimes \sigma^z\right)\right\} \qquad u_\pm, v_\pm \in \mathrm{SU}(2)$$

- Full parameterisation not known for $d > 2$, but: $U^{DU} = (u_+ \otimes u_-)SV(v_+ \otimes v_-)$ ($\sim d^3$ free parameters)

where: $S|i\rangle|j\rangle = |j\rangle|i\rangle$ $\qquad V = \sum_{j=0}^{d-1} \hat{u}^{(j)} \otimes |j\rangle\langle j|$ $\qquad u_\pm, v_\pm, \hat{u}^{(j)} \in \mathrm{SU}(d)$

Dual unitary circuits

A dual unitary circuit is a brickwork circuit composed of dual unitary gates.



Unitary in both temporal and spatial directions.

Exact results

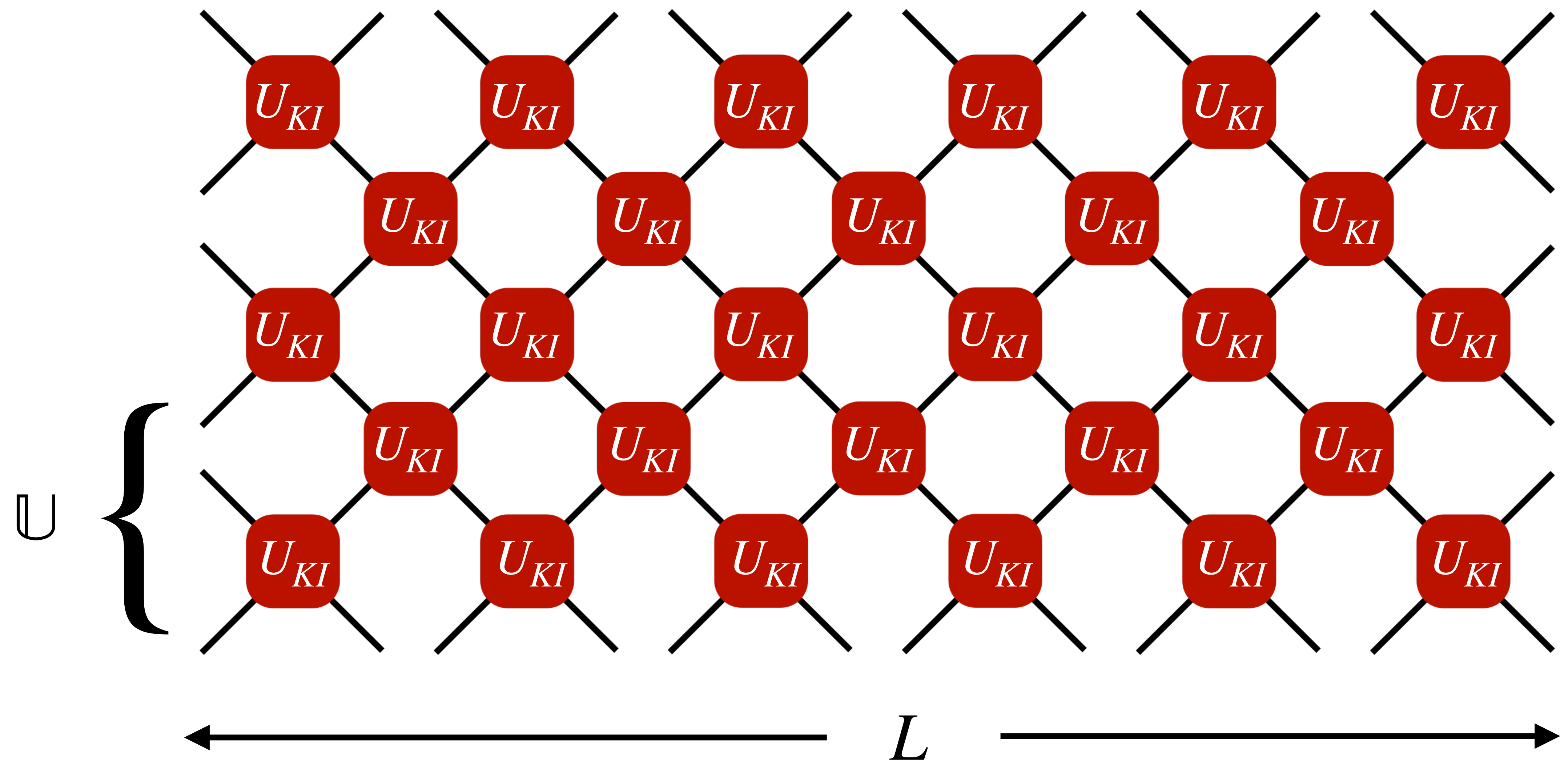
Why are dual unitary circuits interesting? Exact results…

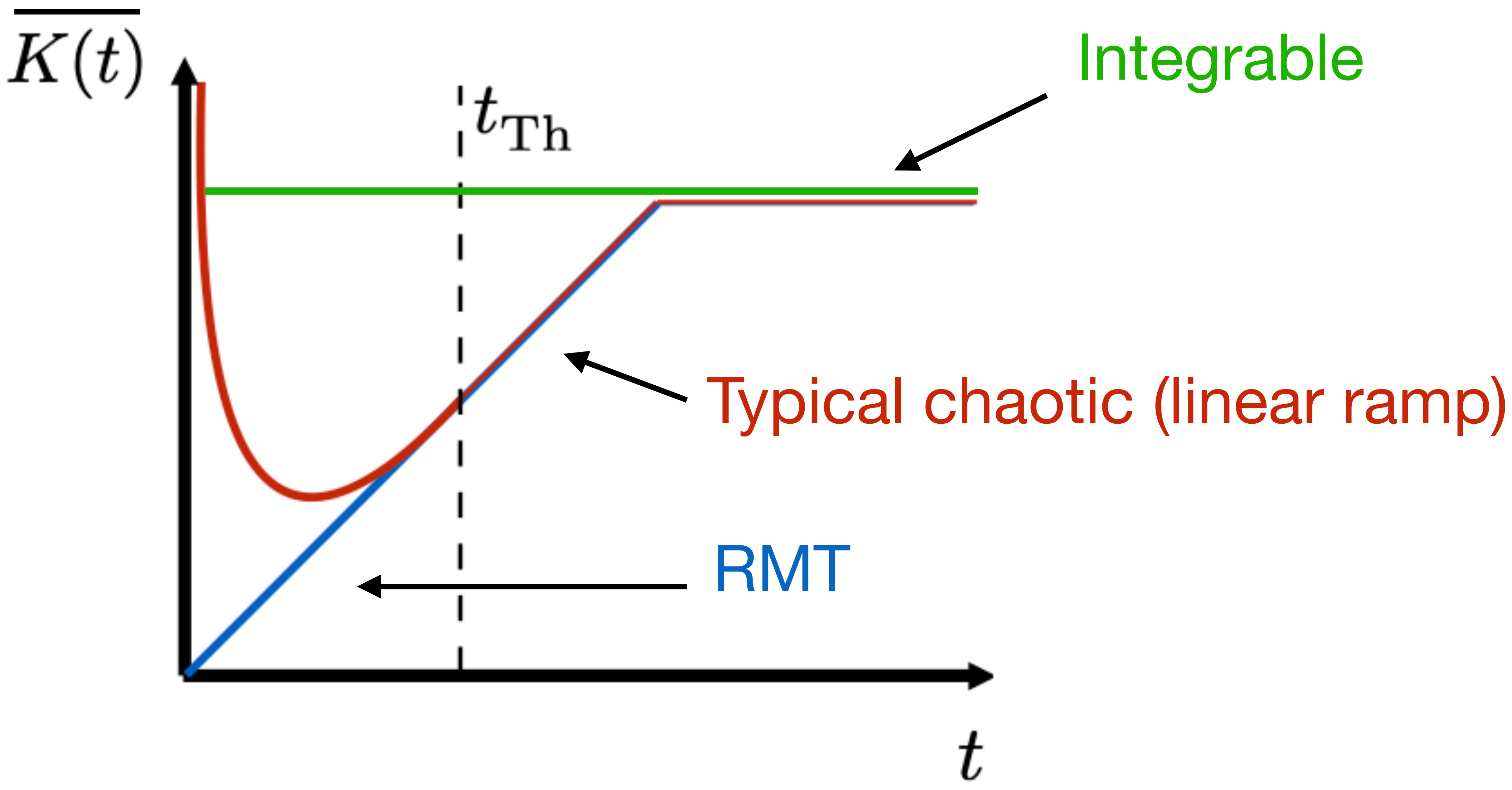
- The dual-unitary kicked Ising model is maximally chaotic [Bertini, Kos, Prosen, PRL (2018)]

$$U_{KI} = V(e^{-i\frac{\pi}{4}\sigma^x} \otimes e^{-i\frac{\pi}{4}\sigma^x})V, \qquad V = e^{-i\frac{\pi}{4}\sigma^z\otimes\sigma^z} e^{-ih_1\sigma^z\otimes\mathbb{I}} e^{-ih_2\mathbb{I}\otimes\sigma^z}$$

- More generally, all (non-swap) $d = 2$ dual-unitary circuits are maximally chaotic [Bertini, Kos, Prosen (2021)] ($\Longrightarrow$ no MBL)

Spectral form factor: $$\lim_{L\to\infty} \overline{K(t)} = \begin{cases} 2t-1, & t \leq 5 \\ 2t, & t \geq 7 \end{cases} \quad (t \text{ odd}), \qquad K(t) = |\mathrm{Tr}(\mathbb{U}^t)|^2$$





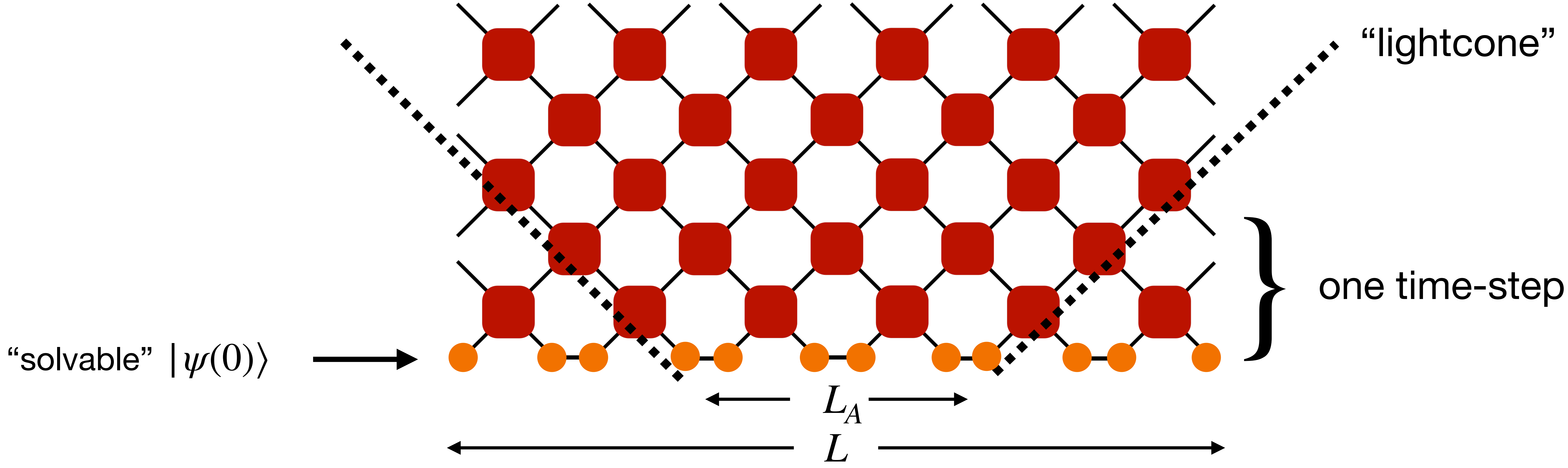
Exact results

- Dual unitary circuits $\Longrightarrow$ fastest entanglement growth (from certain "solvable" initial states)

[Bertini, Kos, Prosen, PRX (2019); Piroli, Bertini, Cirac, Prosen, PRB (2020]

$$\lim_{L\to\infty} S_A^{(\alpha)}(t) = \min(2t, L_A)\log d \qquad S_A^{(\alpha)}(t) = \frac{1}{1-\alpha}\log \mathrm{Tr}([\rho_A(t)]^\alpha)$$

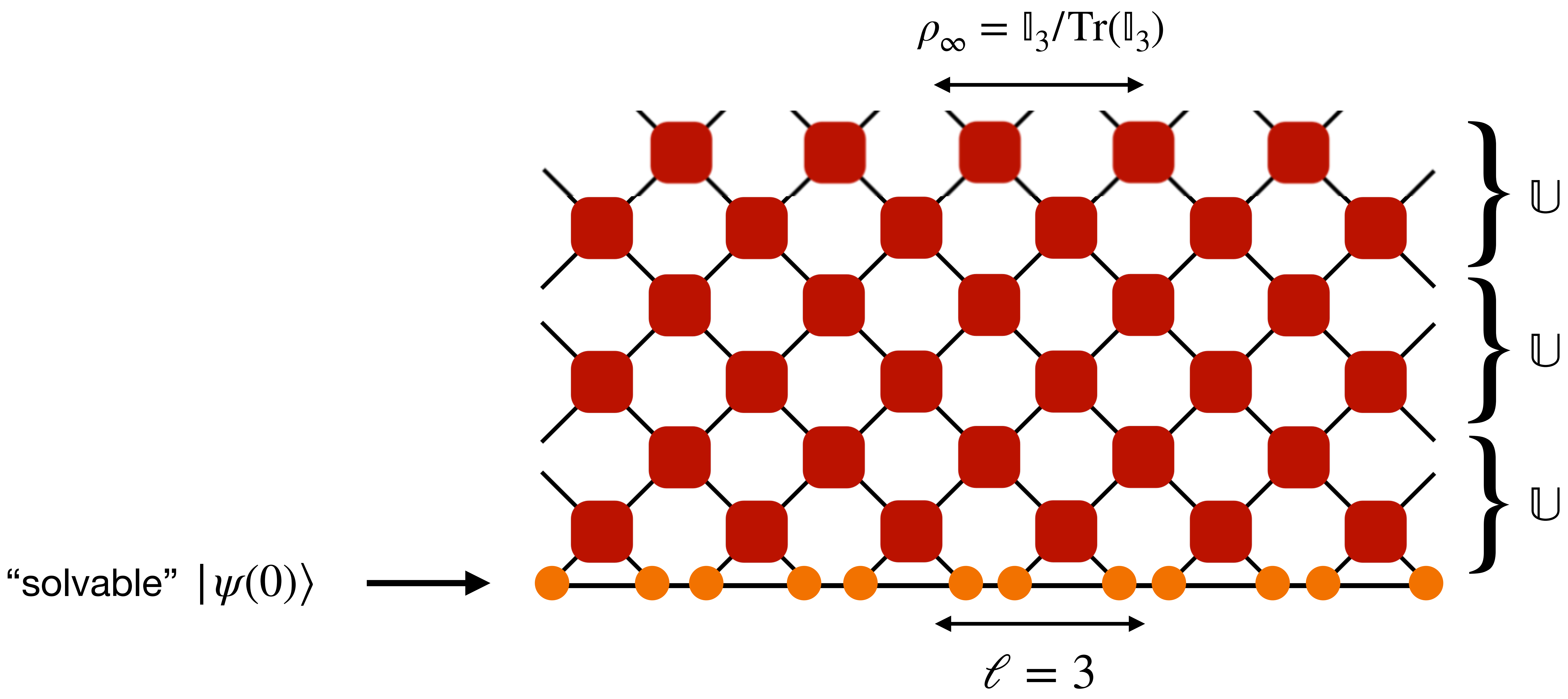
- Fastest entanglement growth $\Longrightarrow$ dual-unitary circuit [Zhou, Harrow (2022)]



Exact results

- Dual unitary circuits $\Longrightarrow$ fast "thermalisation" (from "solvable" initial states)

Any subsystem of size ℓ reaches the infinite-temperature state $\rho_\infty = \mathbb{I}_\ell/\mathrm{Tr}(\mathbb{I}_\ell)$ after a finite time $t \propto \ell$



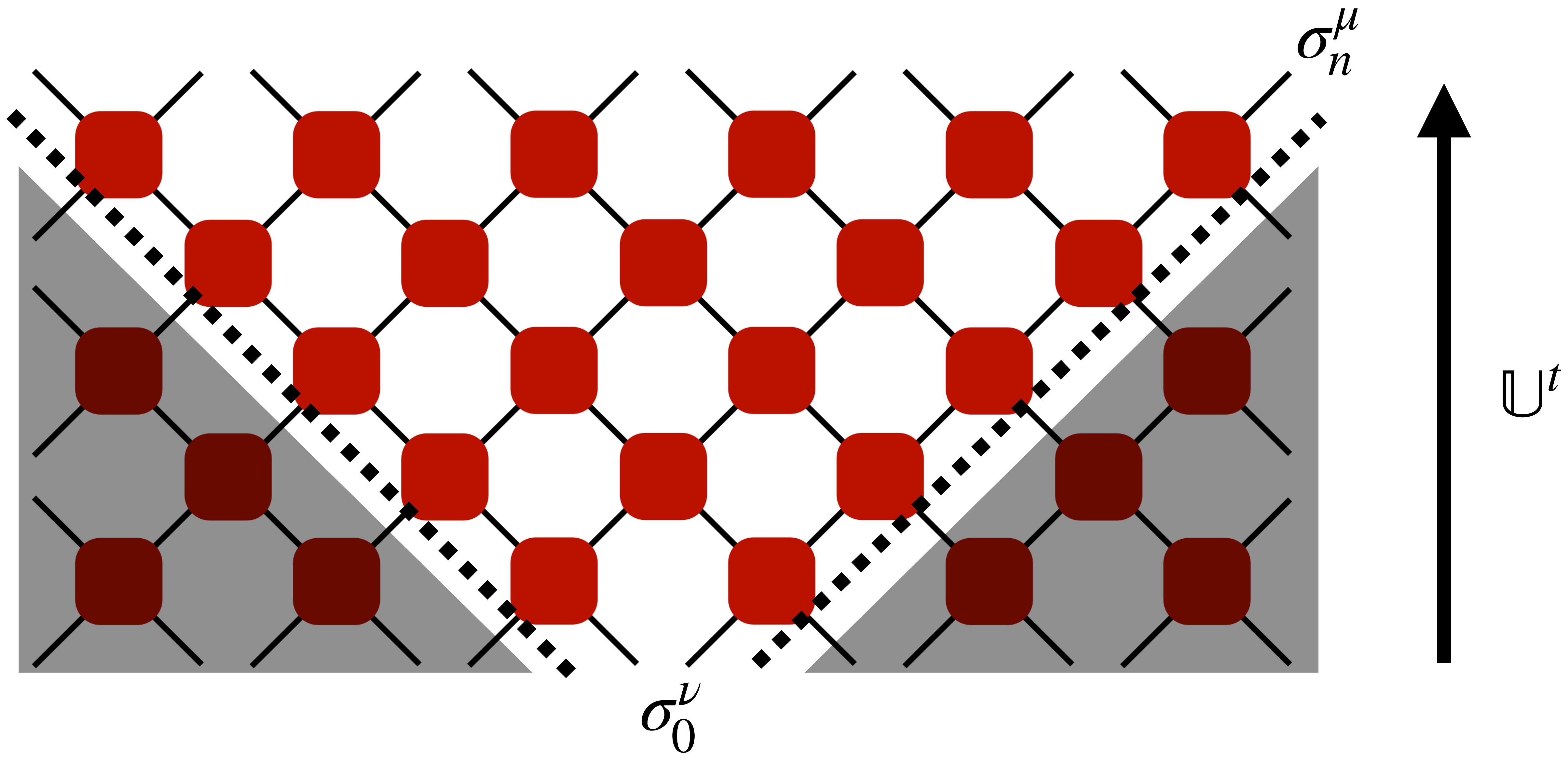
Exact results

- Dual unitary circuits $\Longrightarrow$ fastest spreading of dynamical correlations

$$C_n(t) = \frac{1}{d^L}\mathrm{Tr}[\mathbb{U}^{-t}\sigma_n^{\mu}\mathbb{U}^{t}\sigma_0^{\nu}] \propto \delta_{n,\pm t}$$

[Bertini, Kos, Prosen, PRL (2019)]

Lightcone argument:



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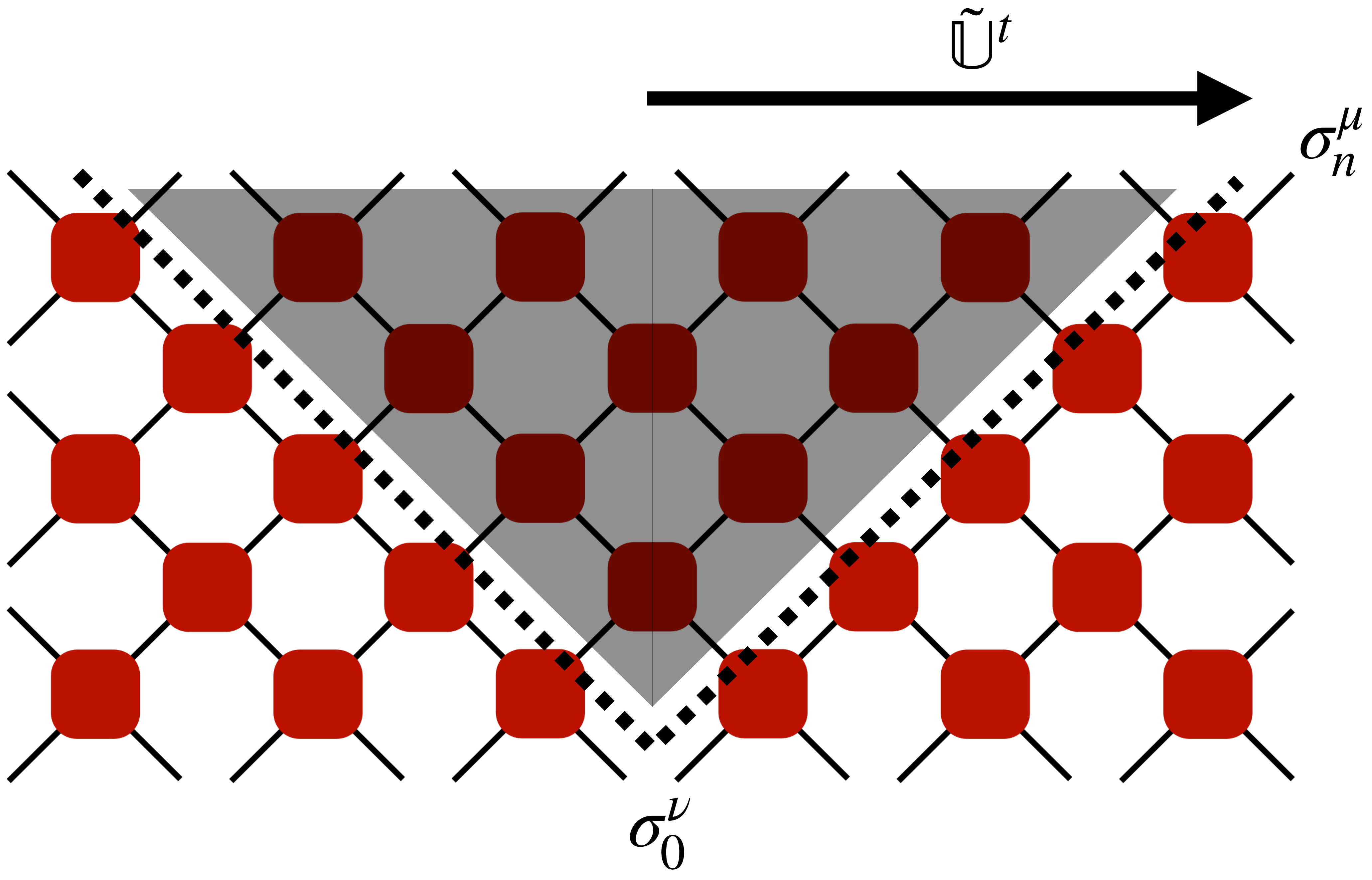

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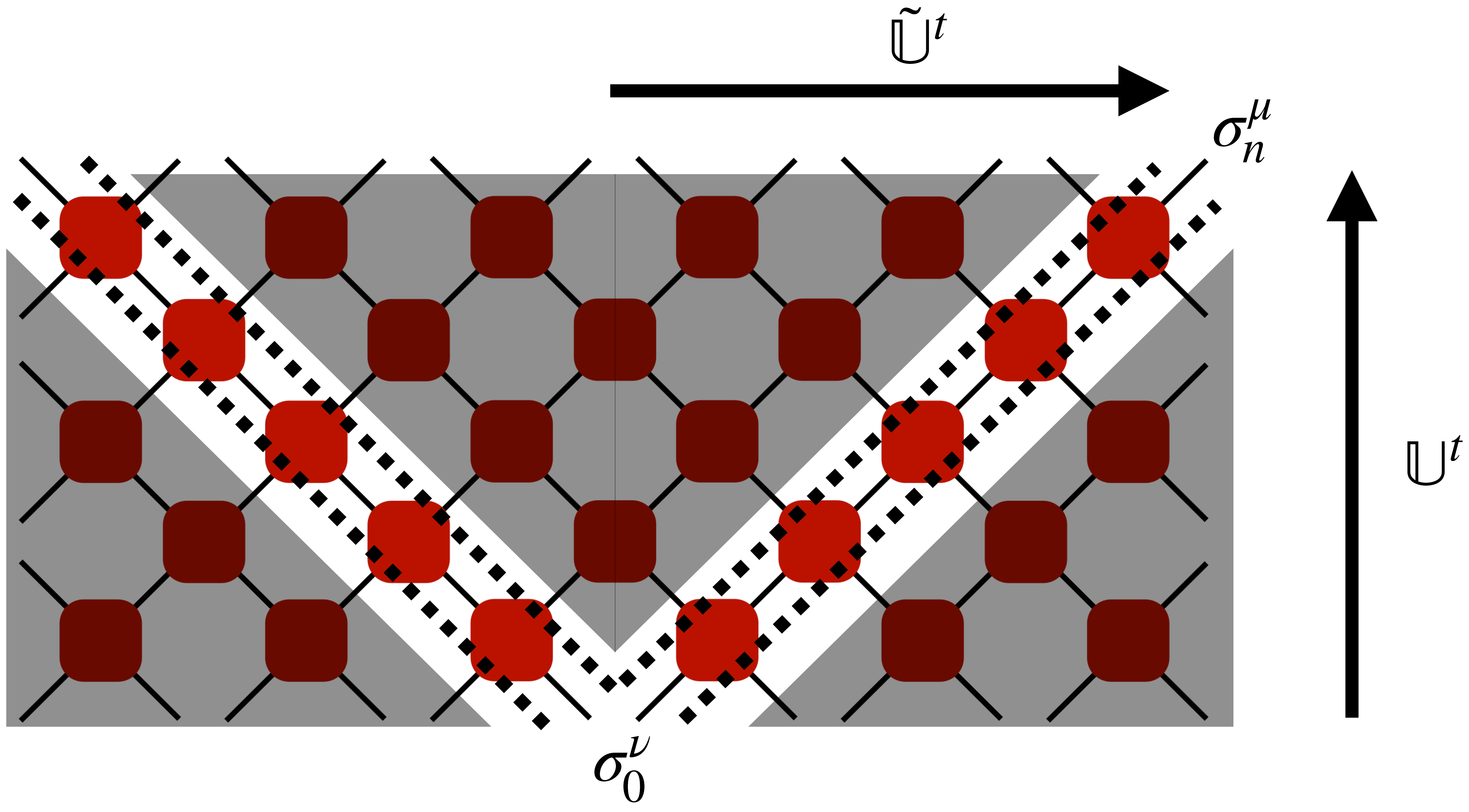
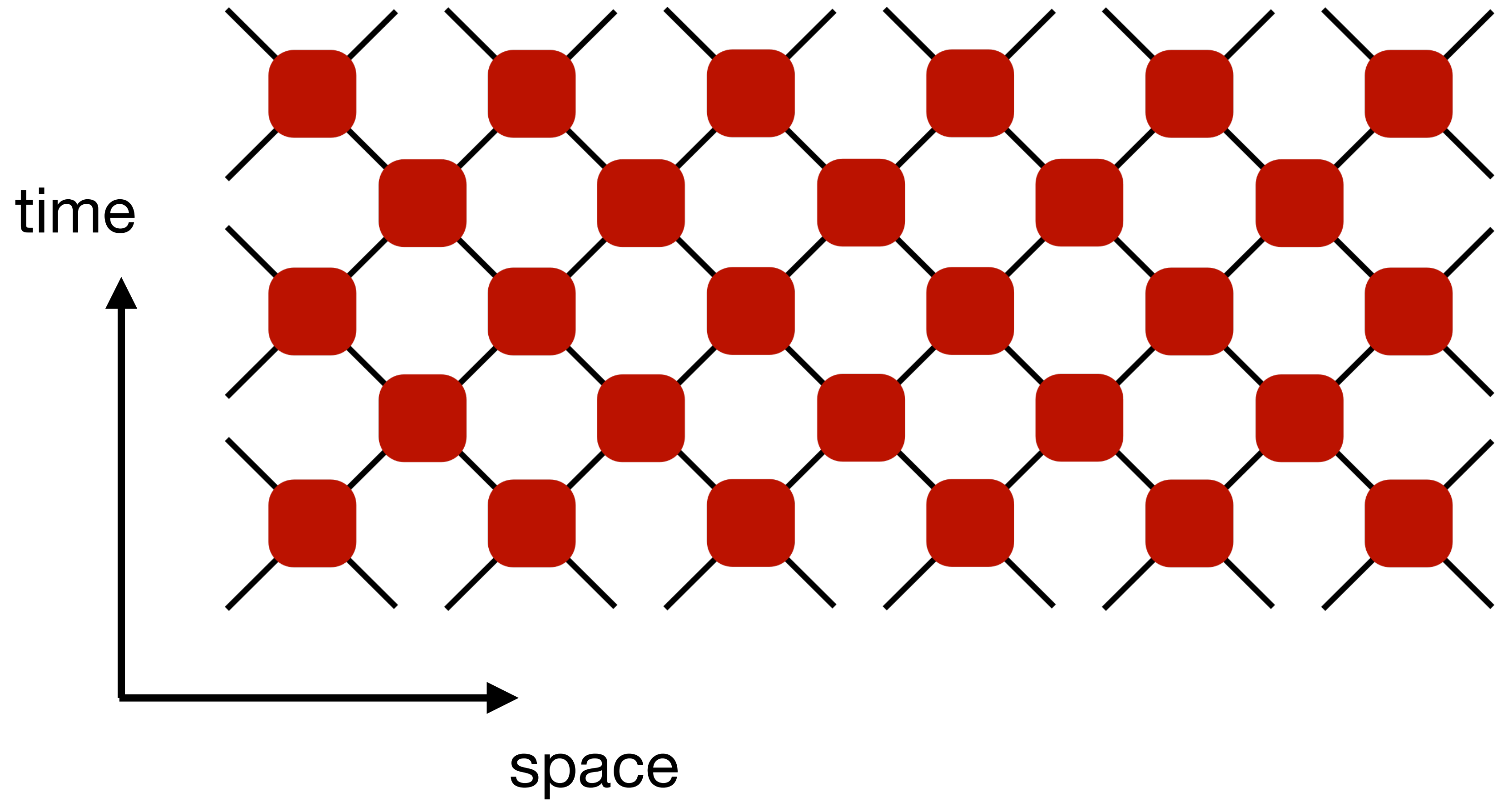


- Exact value of $C_n(t)$ on the lightcone also calculated → used to group DU circuits into ergodic classes

Recap on dual-unitary circuits…

- DU circuits are unitary in both spatial and temporal directions
- This property makes it possible to do some exact calculations (despite nonintegrability), e.g.,
  - “Maximally chaotic”
  - Fast scramblers of quantum information
  - Fast entanglers (from “solvable” initial states)
  - Rapid thermalisation (from “solvable” initial states)
  - No many-body localisation (MBL) through disorder
  - Rigorous classification in terms of ergodic/mixing properties (through dynamical correlation functions)



- All exact results seem to suggest that DU circuits are **very effective thermalising systems**
- Question: **is it possible to *avoid* thermalisation in “maximally chaotic” dual unitary circuits**?
- This talk: yes, through **quantum many-body scars** (QMBS)

3. Embedding QMBS in DU circuits

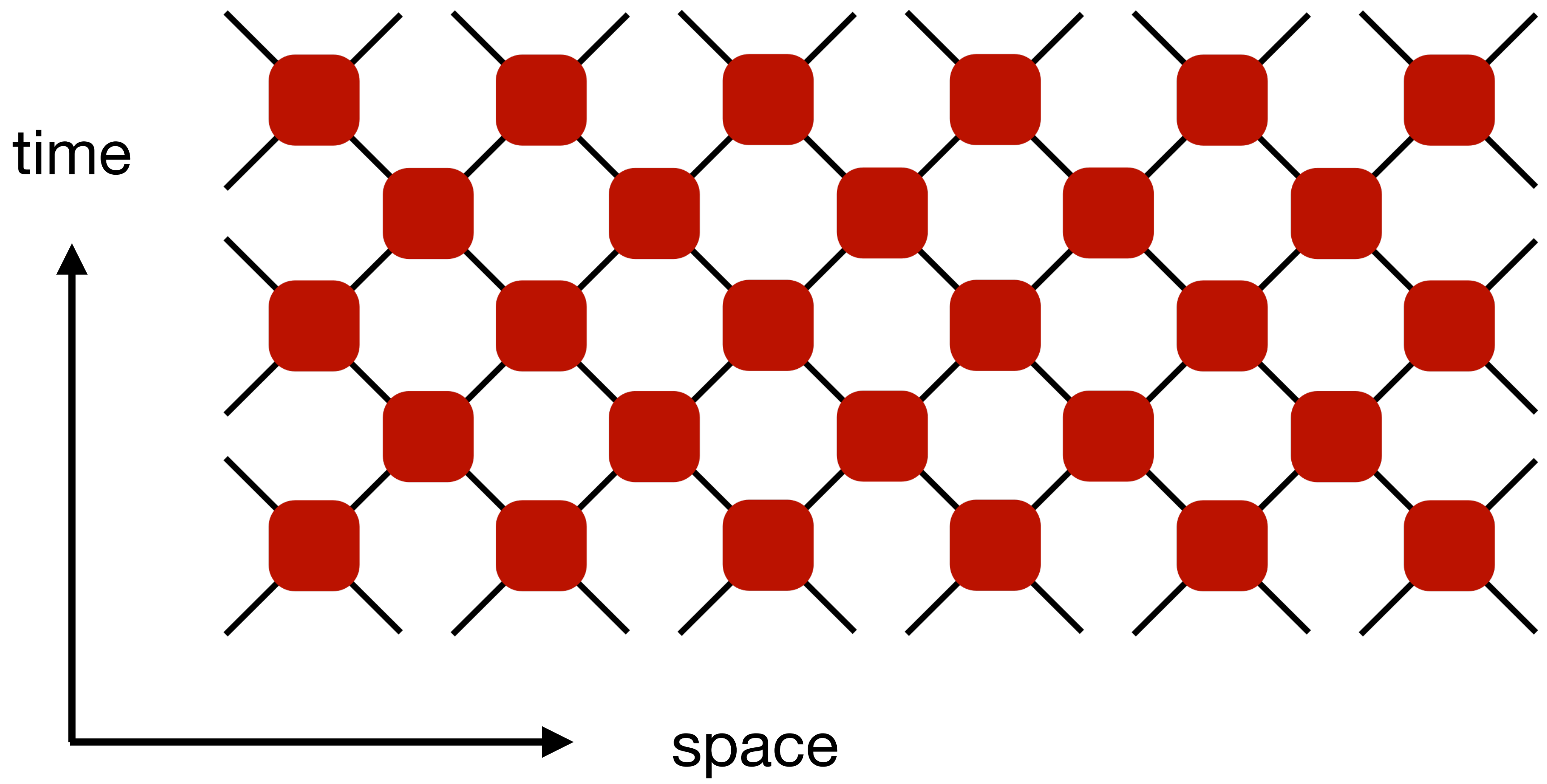
QMBS in DU a circuit?

- Can we have QMBS in a DU circuit that is **provably "maximally chaotic"**?
- Setting: Circuit of N qudits (labelled $n = 0,1,\ldots,N-1$), local basis $\{\,|j\rangle\}_{j=0}^{d-1}$
- Strategy:
  1. Construct a parameterisation for dual-unitary circuits

     [will find one specified by a set of $d \times d$ Hermitian matrices $\{\hat{f}^{\pm}, \hat{g}^{\pm}, \hat{h}^{(j)}\}_{j=0}^{d-1}$]
  2. "Embed" quantum many-body scars (without breaking dual-unitarity)

     [via three conditions C1, C2, C3 on the matrices $\{\hat{f}^{\pm}, \hat{g}^{\pm}, \hat{h}^{(j)}\}$]

     [construction inspired by Shiraishi & Mori, PRL, 119, 030601, (2017)]



Dual-unitary parameterisation

- Two-qudit gates: $\hat{U}^{\mathrm{DU},1} = (\hat{u}^+ \otimes \hat{u}^-)\hat{S}\hat{V}(\hat{v}^- \otimes \hat{v}^+)$ $\quad$ $\hat{U}^{\mathrm{DU},2} = \hat{S}\hat{U}^{\mathrm{DU},1}\hat{S}$

[Prosen, 2021]

Single qudit rotations

$$\hat{u}^+ \otimes \hat{u}^- = \exp\{i(\hat{f}^+ \otimes \hat{\mathbb{I}} + \hat{\mathbb{I}} \otimes \hat{f}^-)\}$$

$$\hat{v}^- \otimes \hat{v}^+ = \exp\{i(\hat{g}^- \otimes \hat{\mathbb{I}} + \hat{\mathbb{I}} \otimes \hat{g}^+)\}$$

Swap gate

$$\hat{S}\,|\,i\rangle \otimes |\,j\rangle = |\,j\rangle \otimes |\,i\rangle$$

Entangling gate

$$V = \exp\left\{ i \sum_{j=0}^{d-1} \hat{h}^{(j)} \otimes |j\rangle\langle j| \right\}$$

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Single qudit rotations

$$\hat{u}^+ \otimes \hat{u}^- = \exp\{i(\hat{f}^+ \otimes \hat{\mathbb{1}} + \hat{\mathbb{1}} \otimes \hat{f}^-)\}$$

$$\hat{v}^- \otimes \hat{v}^+ = \exp\{i(\hat{g}^- \otimes \hat{\mathbb{1}} + \hat{\mathbb{1}} \otimes \hat{g}^+)\}$$

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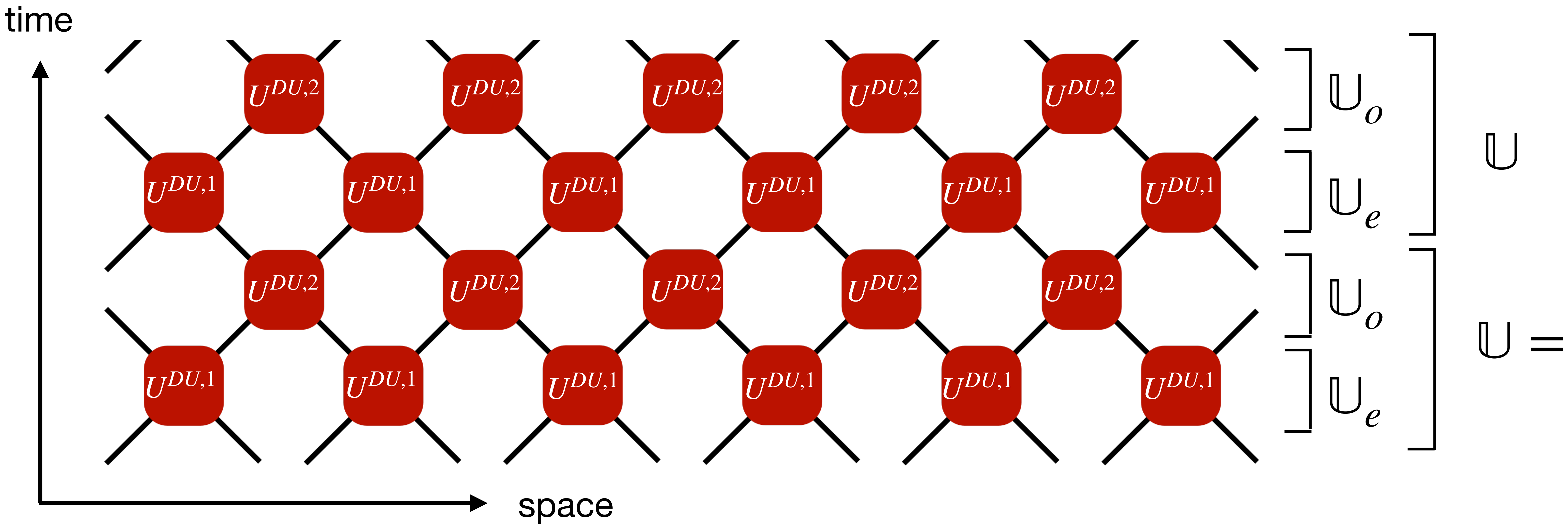

$$|\psi(t)\rangle = \hat{\mathbb{U}}^t |\psi(0)\rangle$$

$$\mathbb{U} = \mathbb{U}_o \mathbb{U}_e$$

- Circuit specified by: $\{\hat{f}^\pm, \hat{g}^\pm, \hat{h}^{(j)}\}$

Embedding QMBS in dual-unitary circuits

- Next…
  - Construct a set of two-qudit projectors: $\hat{P}_{n,n+1} \qquad n = 0,1,...,N-1$
  - Their common kernel is $\mathscr{K} = \{ |\psi\rangle : \hat{\mathbb{P}}_{n,n+1} |\psi\rangle = 0, \forall n \}$ $\qquad \hat{\mathbb{P}}_{n,n+1} = \hat{\mathbb{I}}_{0,n-1} \otimes \hat{P}_{n,n+1} \otimes \hat{\mathbb{I}}_{n+1,N-1}$
  - The subspace of $\mathscr{K}$ that is invariant under layers of swap gates =

$$\mathscr{T} = \{ |\psi\rangle : |\psi\rangle \in \mathscr{K}, \quad \hat{\mathbb{S}}_e |\psi\rangle \in \mathscr{K}, \quad \hat{\mathbb{S}}_o |\psi\rangle \in \mathscr{K} \}$$

= Target set of states that we wish to embed as QMBS

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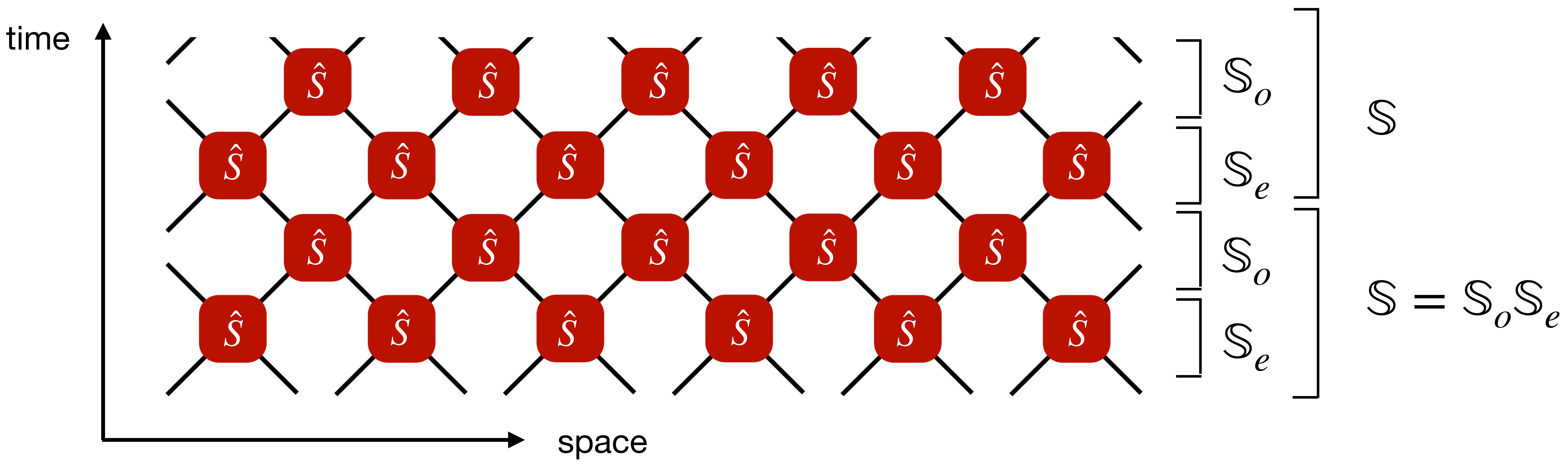

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= Target set of states that we wish to embed as QMBS

- To embed as QMBS: impose three conditions on $\{\hat{f}^{\pm}, \hat{g}^{\pm}, \hat{h}^{(j)}\}$:

C1: $\hat{P}(\hat{f}^+ \otimes \hat{\mathbb{I}} + \hat{\mathbb{I}} \otimes \hat{f}^-)\hat{P} = \hat{f}^+ \otimes \hat{\mathbb{I}} + \hat{\mathbb{I}} \otimes \hat{f}^- \implies \hat{u}^+ \otimes \hat{u}^-|\psi\rangle = |\psi\rangle, \quad |\psi\rangle \in \mathscr{K}$

C2: $\hat{P}(\hat{g}^- \otimes \hat{\mathbb{I}} + \hat{\mathbb{I}} \otimes \hat{g}^+)\hat{P} = \hat{g}^- \otimes \hat{\mathbb{I}} + \hat{\mathbb{I}} \otimes \hat{g}^+ \implies \hat{v}^- \otimes \hat{v}^+|\psi\rangle = |\psi\rangle, \quad |\psi\rangle \in \mathscr{K}$

C3: $\hat{P}\left(\sum_{j=0}^{d-1} \hat{h}^{(j)} \otimes |j\rangle\langle j|\right)\hat{P} = \sum_{j=0}^{d-1} \hat{h}^{(j)} \otimes |j\rangle\langle j| \implies \hat{V}|\psi\rangle = |\psi\rangle, \quad |\psi\rangle \in \mathscr{K}$

- Outcome:
  - Initial states $|\psi(0)\rangle \in \mathscr{T}$ evolve by the elementary swap circuit $|\psi(t)\rangle = \hat{\mathbb{U}}^t|\psi(0)\rangle = \hat{\mathbb{S}}^t|\psi(0)\rangle$
  - Initial states $|\psi(0)\rangle \in \mathscr{T}^{\perp}$ evolve by a more complicated (chaotic) dynamics $|\psi(t)\rangle = \hat{\mathbb{U}}^t|\psi(0)\rangle$

Example A: single QMBS

- Projectors: $\hat{P}_{n,n+1} = \hat{\mathbb{I}}_{n,n+1} - |0\rangle\langle 0|_n \otimes |0\rangle\langle 0|_{n+1}$

- Target QMBS subspace: $\mathscr{K} = \mathscr{T} = \{|0\rangle^{\otimes N}\}$ (since $\hat{\mathbb{S}}_{e/o}|0\rangle^{\otimes N} = |0\rangle^{\otimes N}$)

- Choose $d \times d$ Hermitian matrices $\{\hat{f}^{\pm}, \hat{g}^{\pm}, \hat{h}^{(j)}\}$ randomly, apart from a few matrix elements:

$$\langle i|\hat{f}^{\pm}|0\rangle = \langle i|\hat{g}^{\pm}|0\rangle = \langle i|\hat{h}^{(0)}|0\rangle = 0, \qquad i \in \{0,1,\ldots,d-1\}$$

(to satisfy conditions C1—C3)

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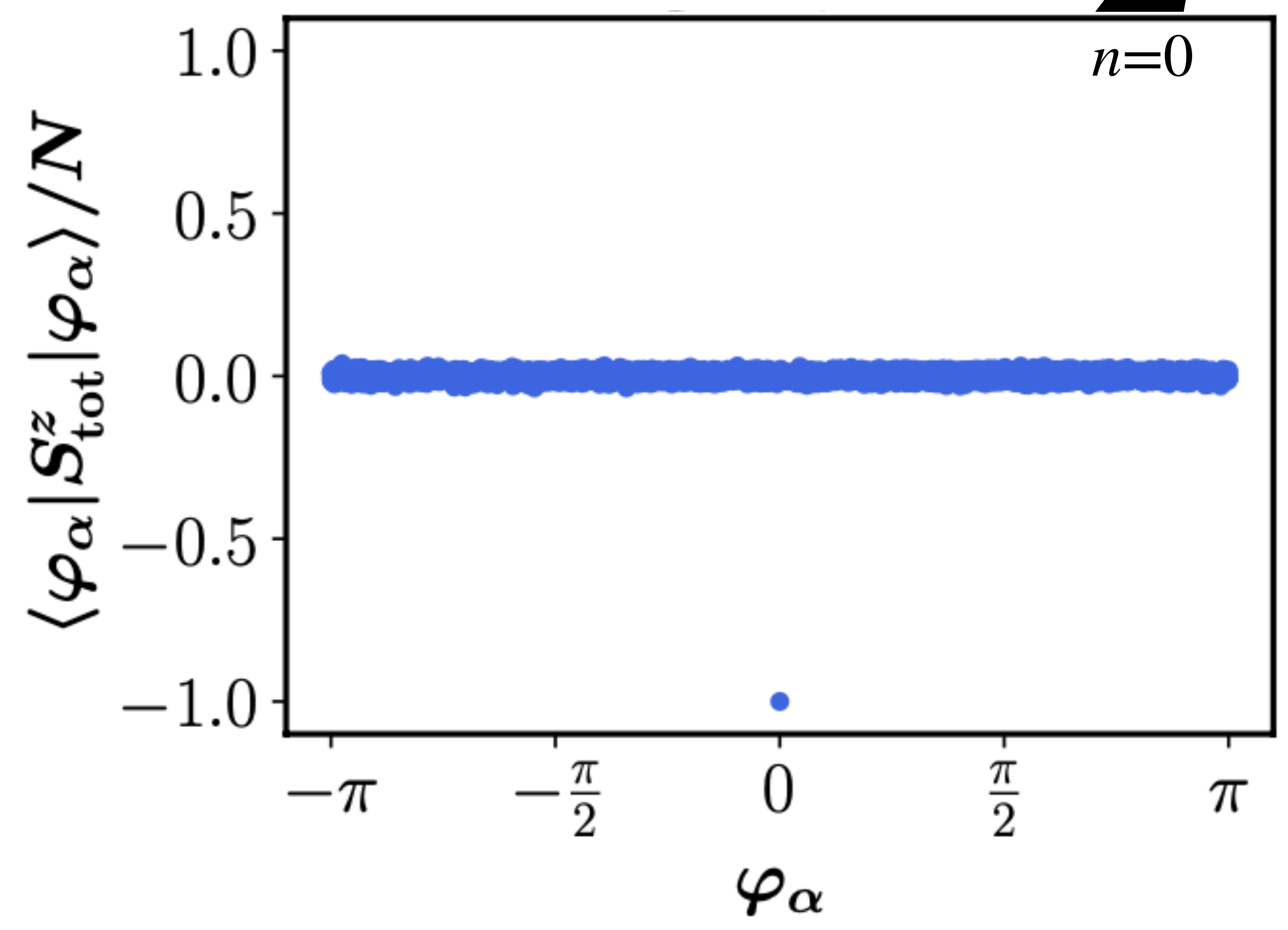
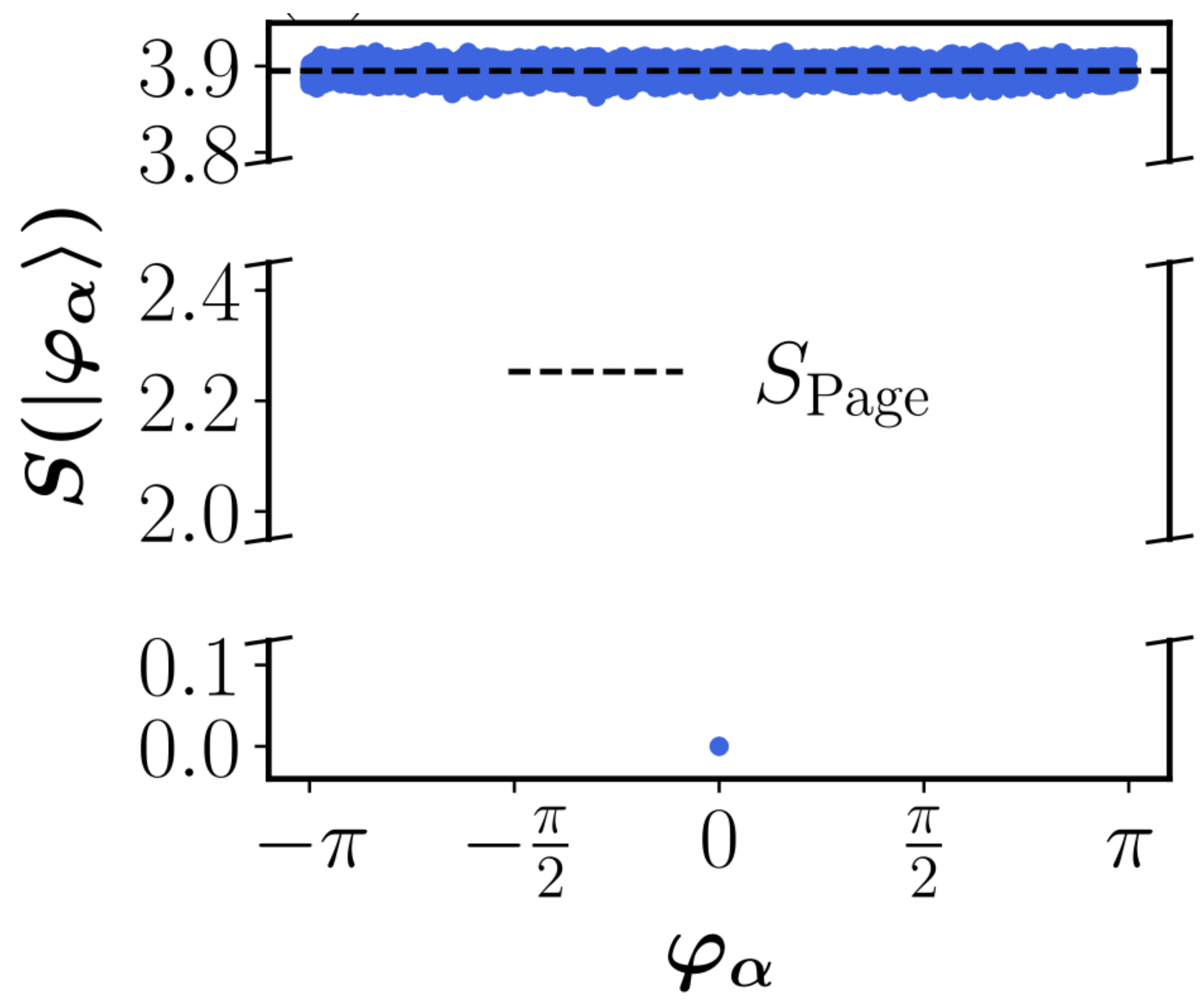
- By construction, circuit is DU and $\hat{\mathbb{U}}|0\rangle^{\otimes N} = |0\rangle^{\otimes N}$ is a QMBS.
- Confirmed numerically…

Diagonal ETH, $\hat{S}^z_{tot} = \sum_{n=0}^{N-1} \hat{S}^z_n$



Eigenstate entanglement



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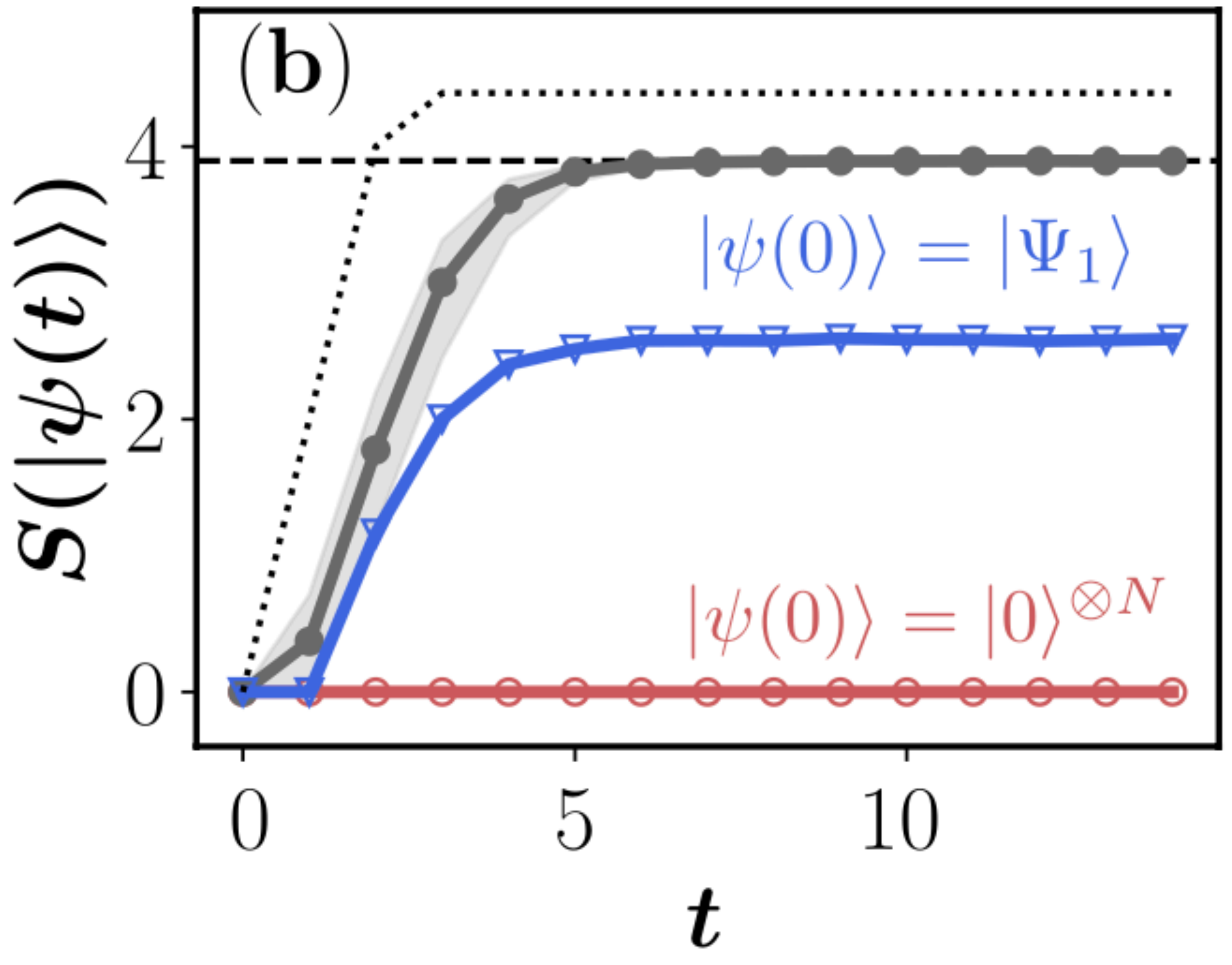
(to satisfy conditions C1—C3)

- Entanglement growth: $S(|\psi(t)\rangle) = -\mathrm{Tr}[\hat{\rho}(t)\ln\hat{\rho}(t)]$

$$\hat{\rho}(t) = \mathrm{Tr}_{0,N/2-1}|\psi(t)\rangle\langle\psi(t)| \qquad |\psi(t)\rangle = \hat{\mathbb{U}}^t|\psi(0)\rangle$$

  - $|0\rangle^{\otimes N}$ QMBS (no entanglement growth)
  - $|j_0, j_1, \ldots, j_{d-1}\rangle$ Random product (rapid ent. growth)
  - $|\Psi_1\rangle = |0\rangle^{\otimes N-1} \otimes (|0\rangle + |d-1\rangle)/\sqrt{2}$

    Equal superposition of QMBS and non-QMBS



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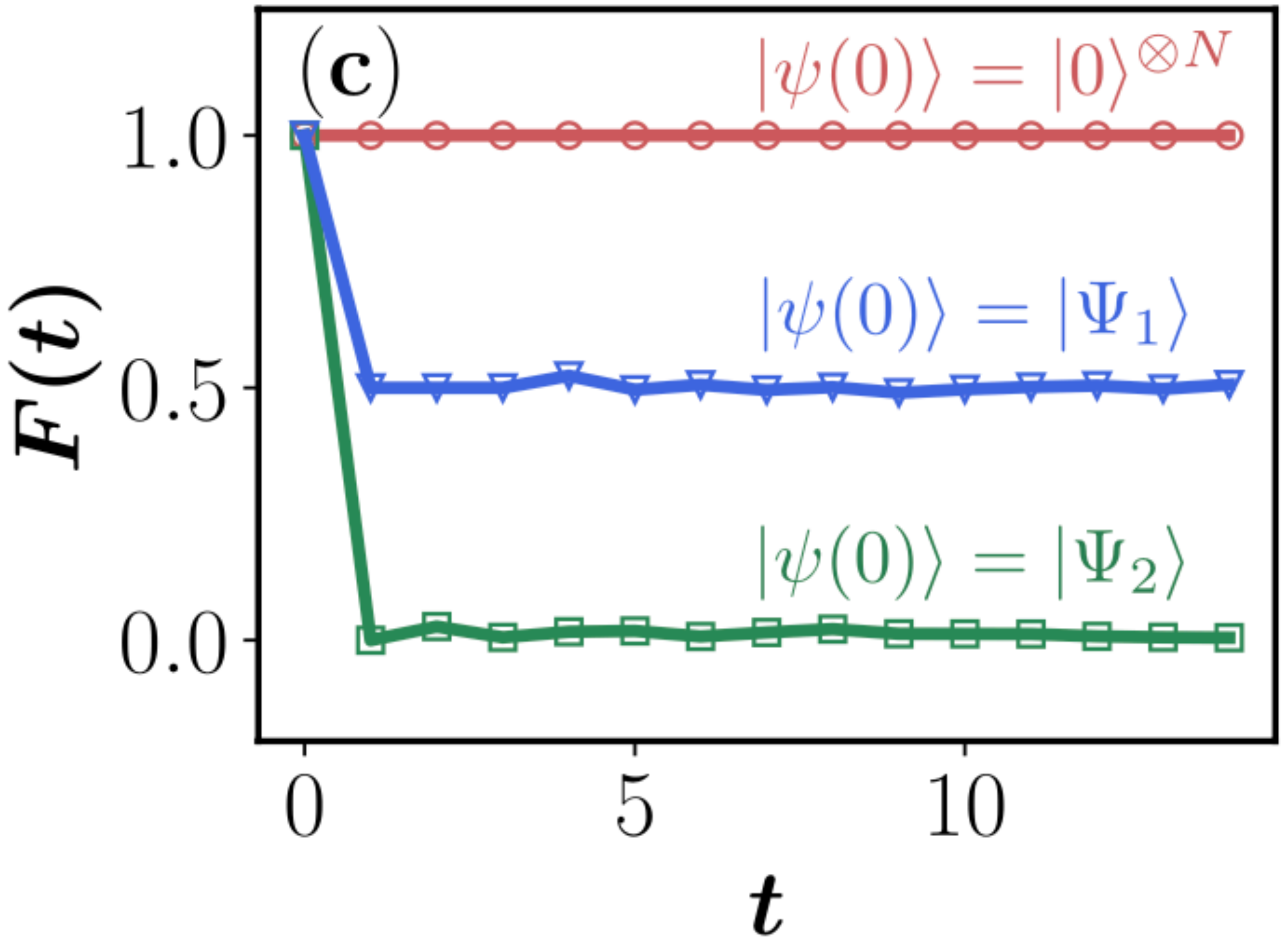
(to satisfy conditions C1—C3)

- Loschmidt echo: $F(t) = |\langle\psi(0)|\psi(t)\rangle|$
  - $|0\rangle^{\otimes N}$ QMBS (no fidelity decay)
  - $|\Psi_1\rangle = |0\rangle^{\otimes N-1} \otimes (|0\rangle + |d-1\rangle)/\sqrt{2}$

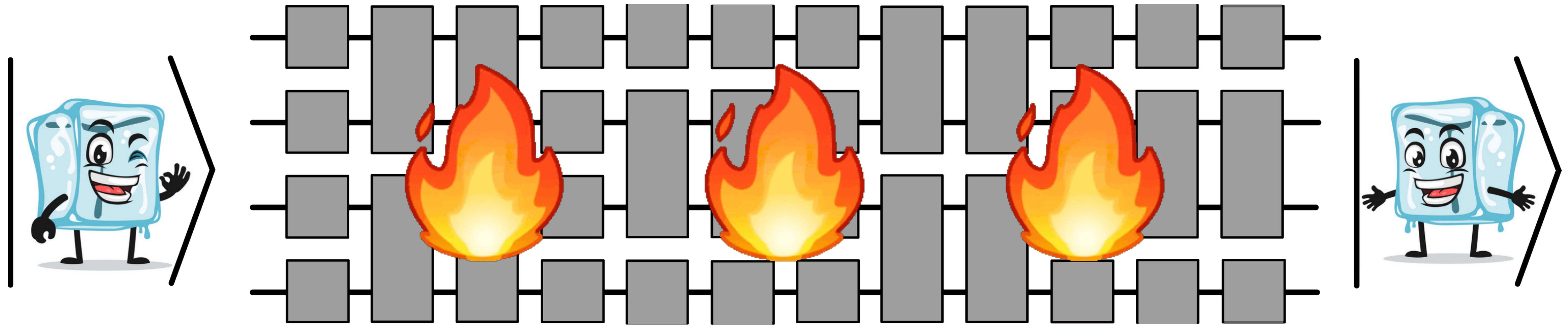
    Equal superposition of QMBS and non-QMBS
  - $|\Psi_2\rangle = |0,0,d-1,d-1\rangle^{\otimes N/4-1} \otimes |0,0,d-1\rangle \otimes (|0\rangle + |d-1\rangle)/\sqrt{2}$

    Non-QMBS



Summary

- The class of quantum system called **dual-unitary circuits** provide rare examples of chaotic many-body systems with exactly solvable quantities.
- Exact results suggest that they are **very efficient thermalising** systems
    - “Maximally chaotic”
    - Fast scramblers of quantum information
    - Fast entanglers (from “solvable” initial states)
    - Rapid thermalisation (from “solvable” initial states)
    - No many-body localisation (MBL) through disorder
- *Provably* chaotic
- Despite this we can find **simple initial states that fail to thermalise** in such systems
    - Achieved by embedding **QMBS** [Phys. Rev. Lett. 132, 010401 (2024)]



Additional slides

Example B: Exponentially many QMBS

- Projectors: $\hat{P}_{n,n+1} = \hat{\mathbb{I}}_{n,n+1} - |0\rangle\langle 0|_n \otimes |0\rangle\langle 0|_{n+1} - |d-1\rangle\langle 0|_n \otimes |0\rangle\langle d-1|_{n+1}$
  $- |0\rangle\langle d-1|_n \otimes |d-1\rangle\langle 0|_{n+1} - |d-1\rangle\langle d-1|_n \otimes |d-1\rangle\langle d-1|_{n+1}$
- Target QMBS subspace: $\mathscr{K} = \mathscr{T} = \{\, |j_0, j_1, \ldots, j_{N-1}\rangle\}_{j_n \in \{0, d-1\}}$
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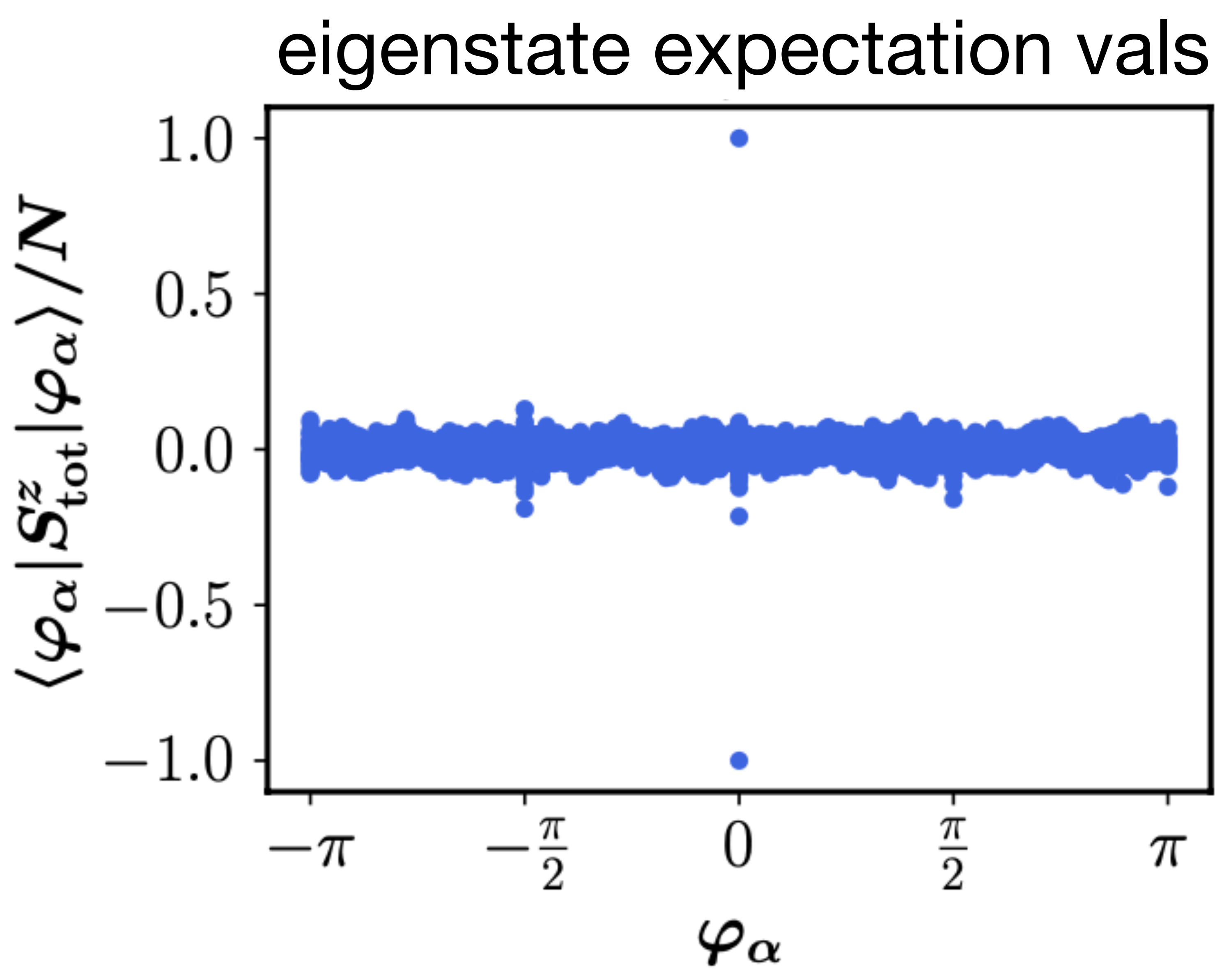

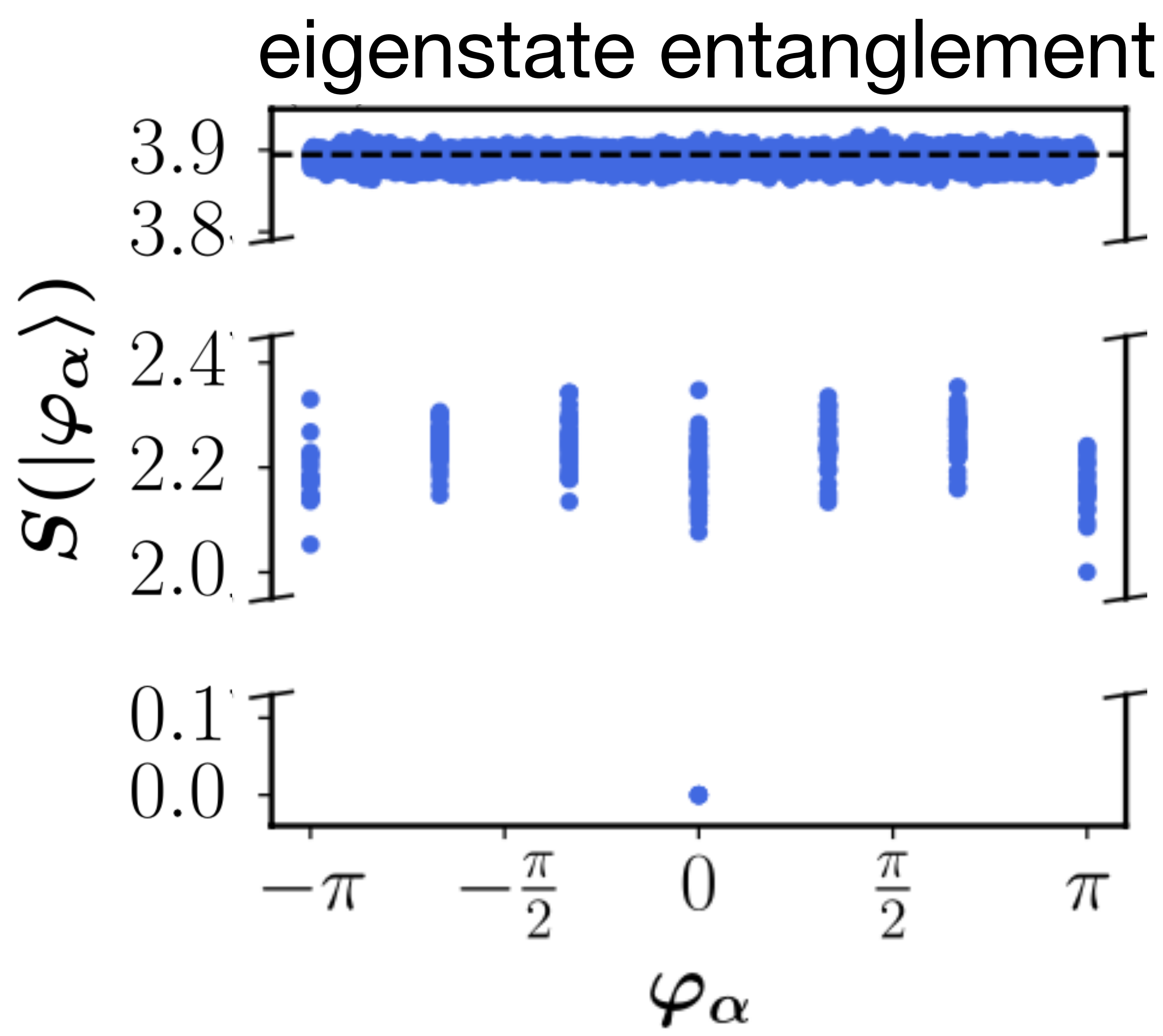

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$$- |0\rangle\langle d-1|_n \otimes |d-1\rangle\langle 0|_{n+1} - |d-1\rangle\langle d-1|_n \otimes |d-1\rangle\langle d-1|_{n+1}$$

- Target QMBS subspace: $\mathscr{K} = \mathscr{T} = \{|j_0, j_1, \ldots, j_{N-1}\rangle\}_{j_n \in \{0,d-1\}}$

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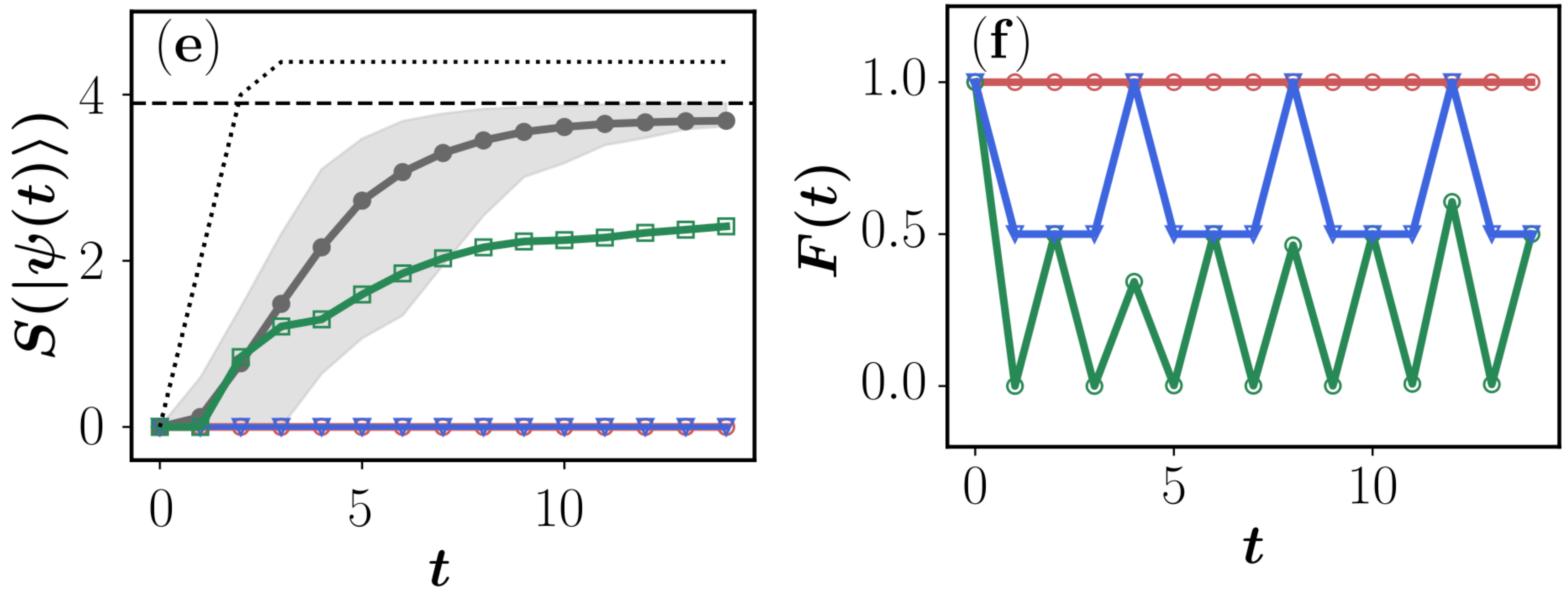
$$\hat{\mathbb{U}}^t |\psi(0)\rangle = \hat{\mathbb{S}}^t |\psi(0)\rangle \quad \text{for} \quad |\psi(0)\rangle \in \mathscr{T}$$

entanglement growth

fidelity decay

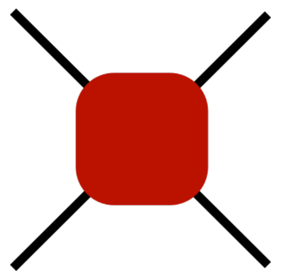
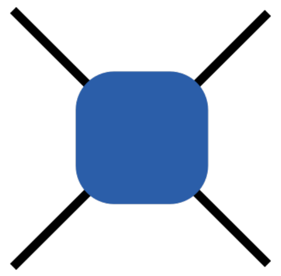


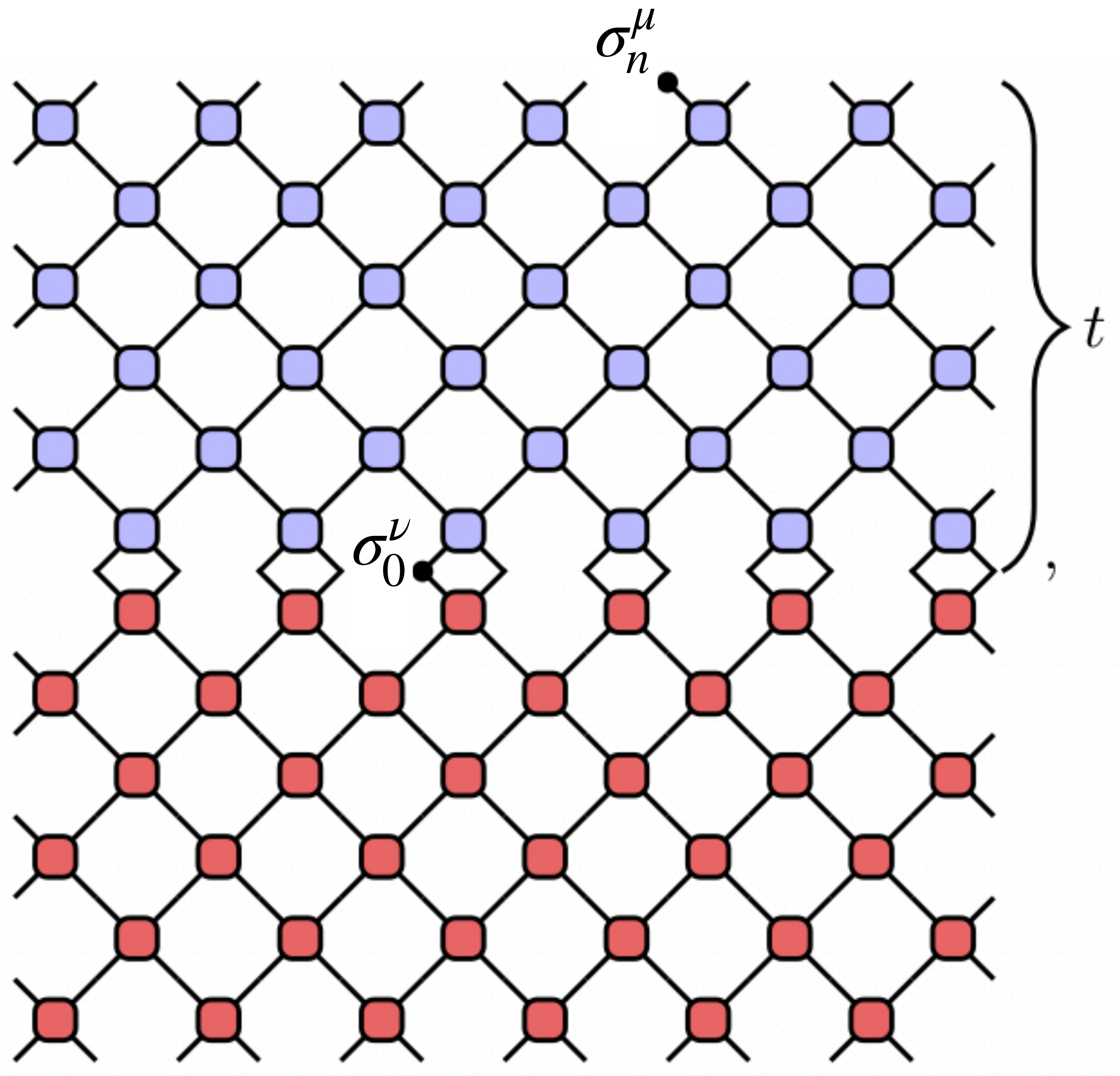
Exact results

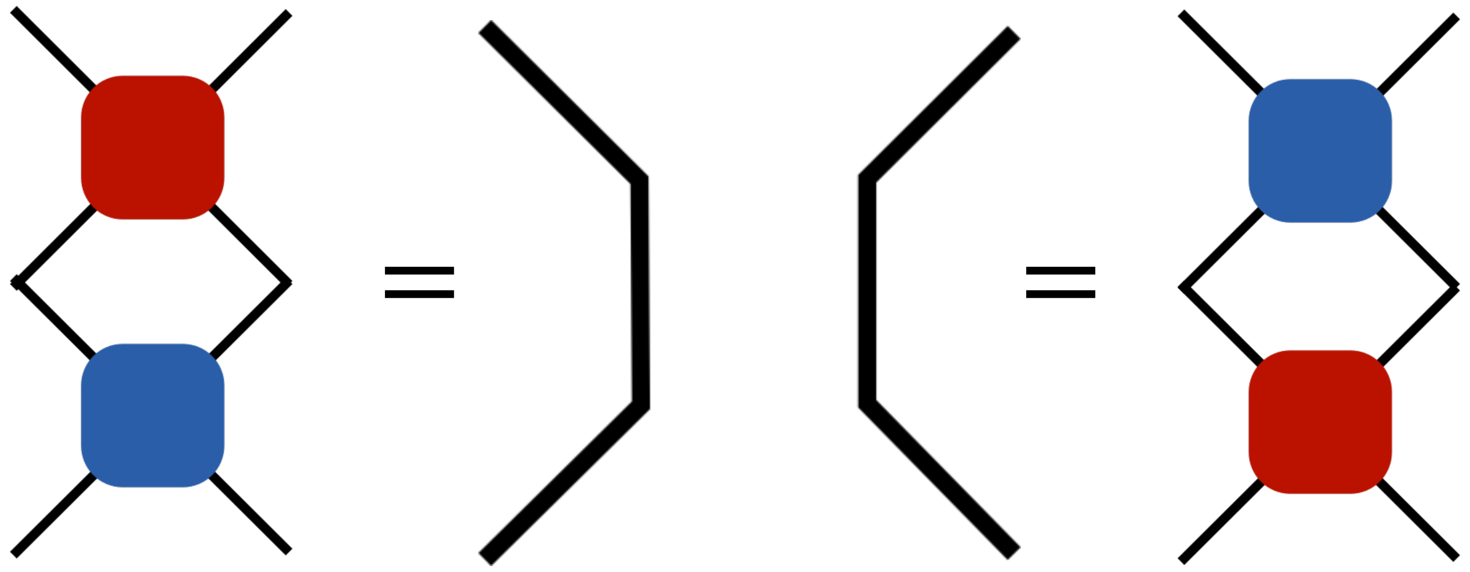
- Dual unitary circuits $\Longrightarrow$ fastest spreading of dynamical correlations

$$C_n(t) = \frac{1}{d^L}\mathrm{Tr}[\mathbb{U}^{-t}\sigma_n^{\mu}\mathbb{U}^{t}\sigma_0^{\nu}] \propto \delta_{n,\pm t}$$

[Bertini, Kos, Prosen, PRL (2019)]

More formally, use unitarity and the diagrammatic tensor notation: U :  $U^{\dagger}$: 

$C_n(t) \propto$ 

unitarity: $UU^{\dagger}$:  : $U^{\dagger}U$

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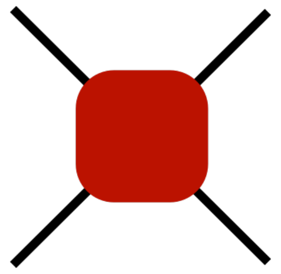
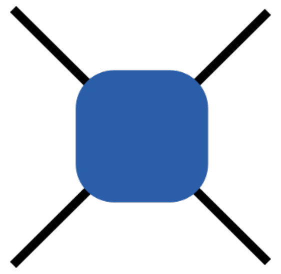
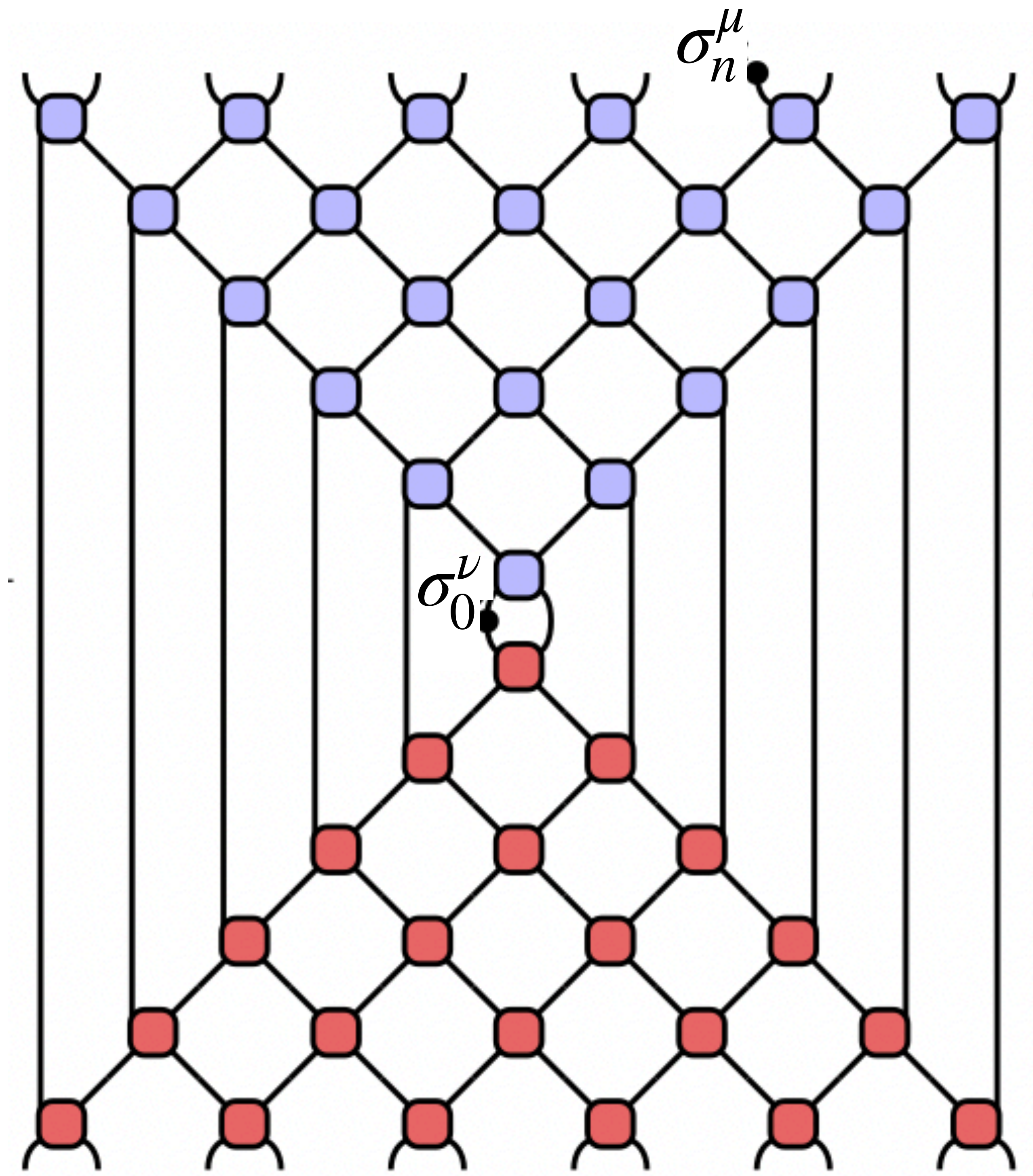
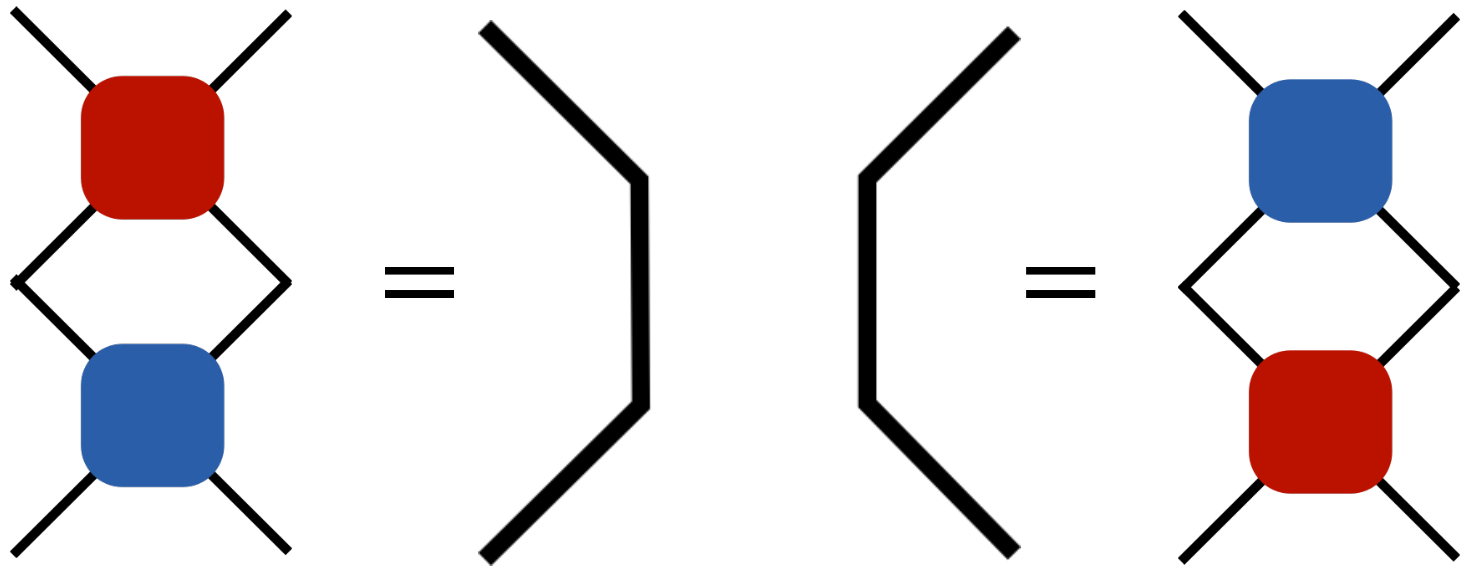
More formally, use unitarity and the diagrammatic tensor notation: U :  $U^\dagger$: 

$C_n(t) \propto$ 

unitarity: $UU^\dagger$:  : $U^\dagger U$

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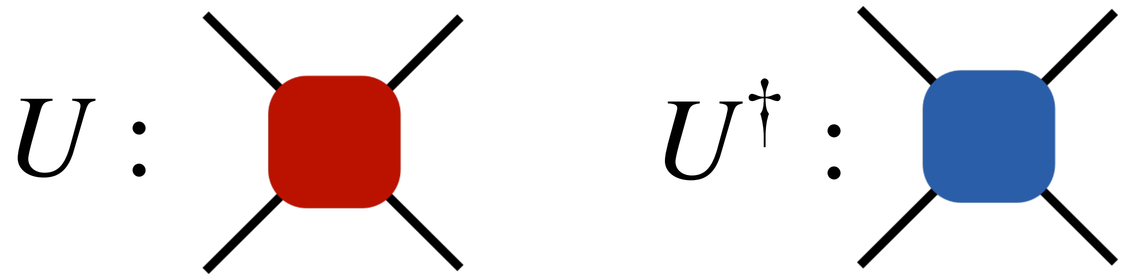

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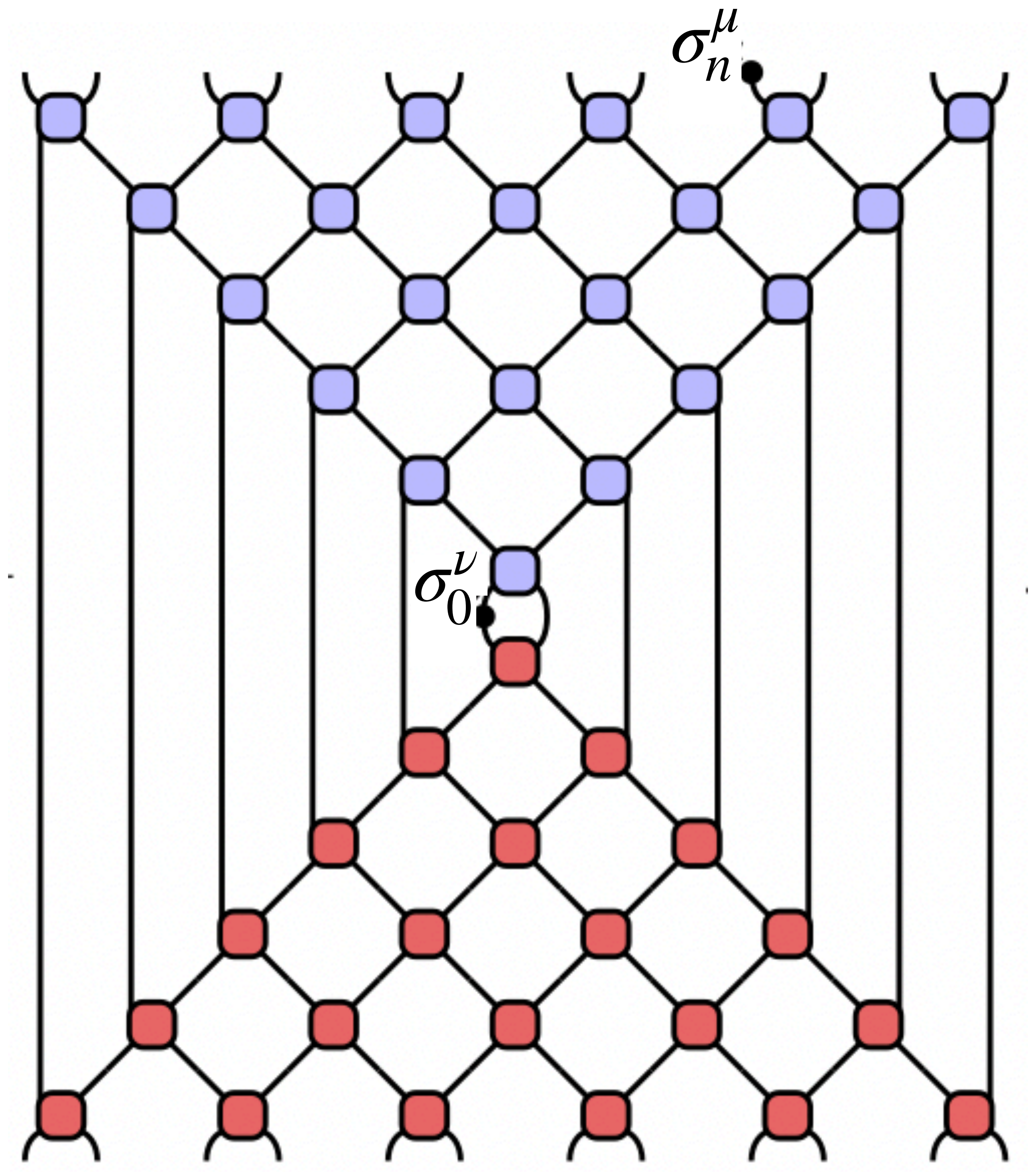
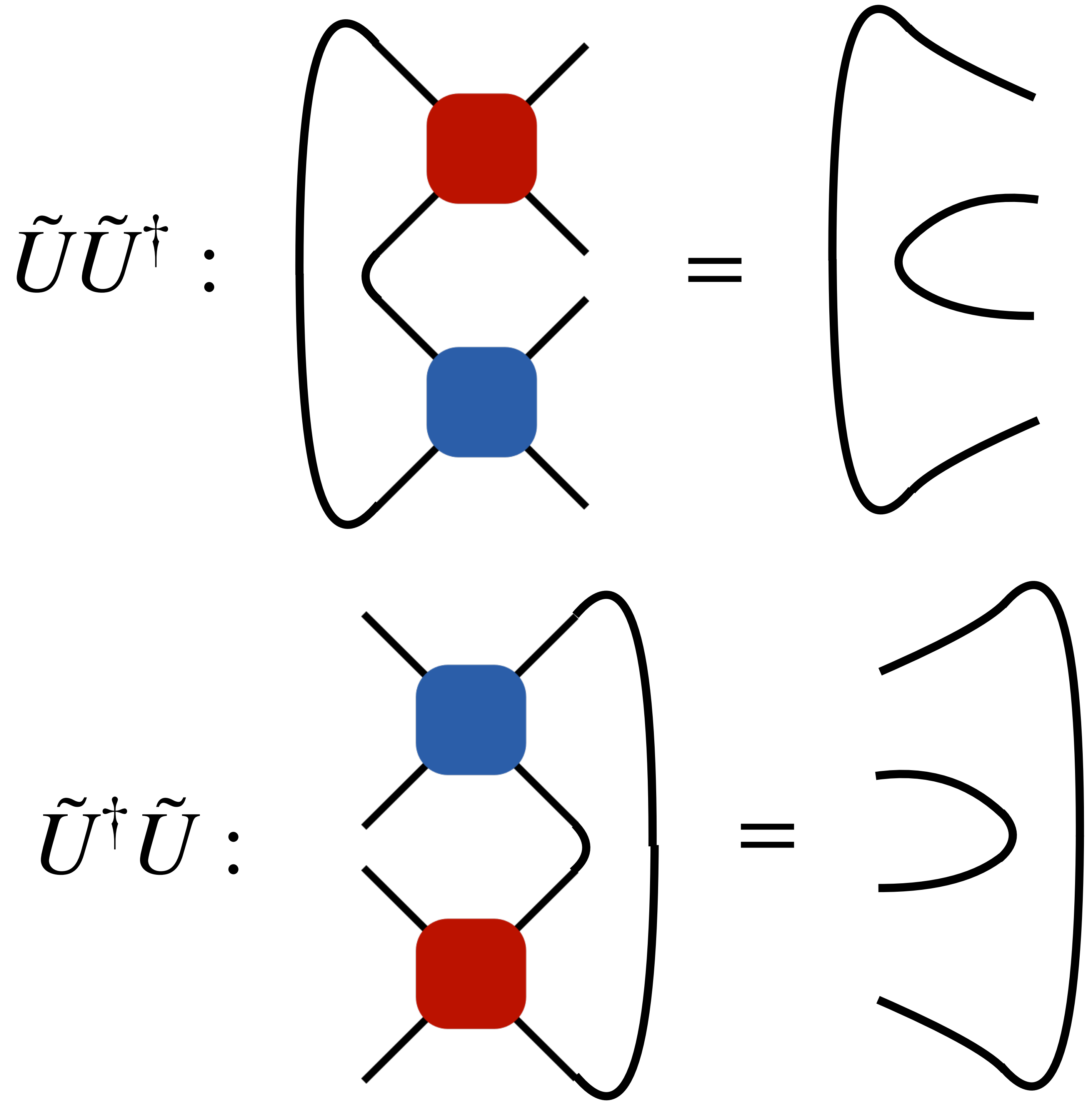


dual unitarity:



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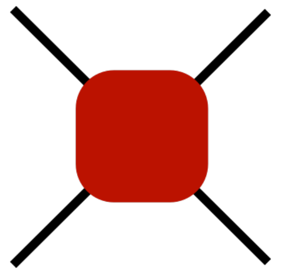
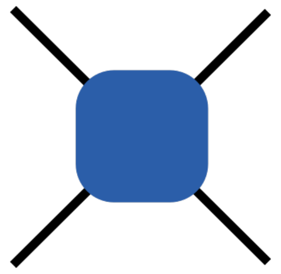
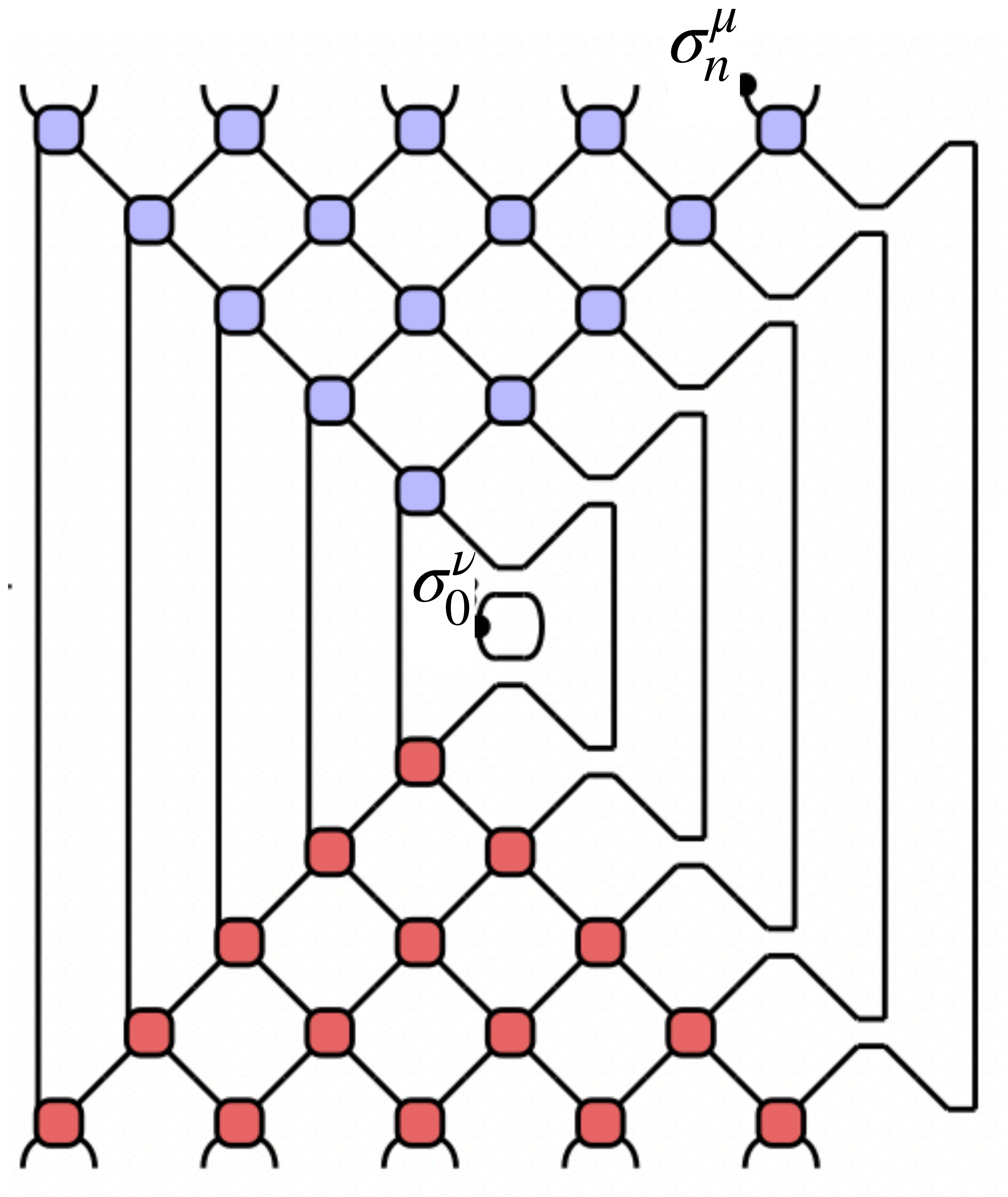
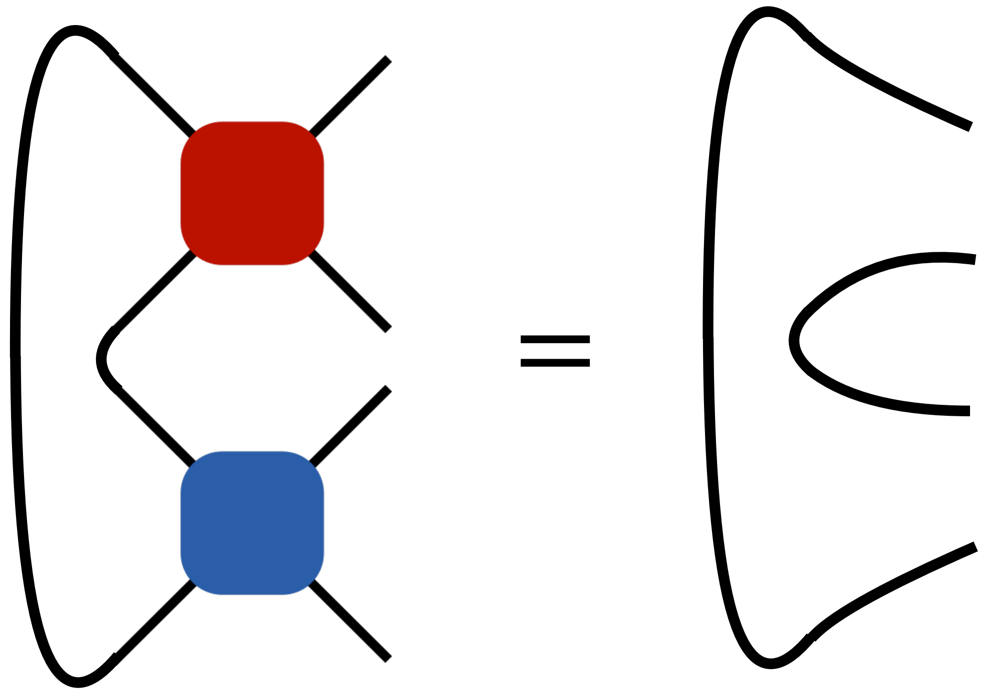
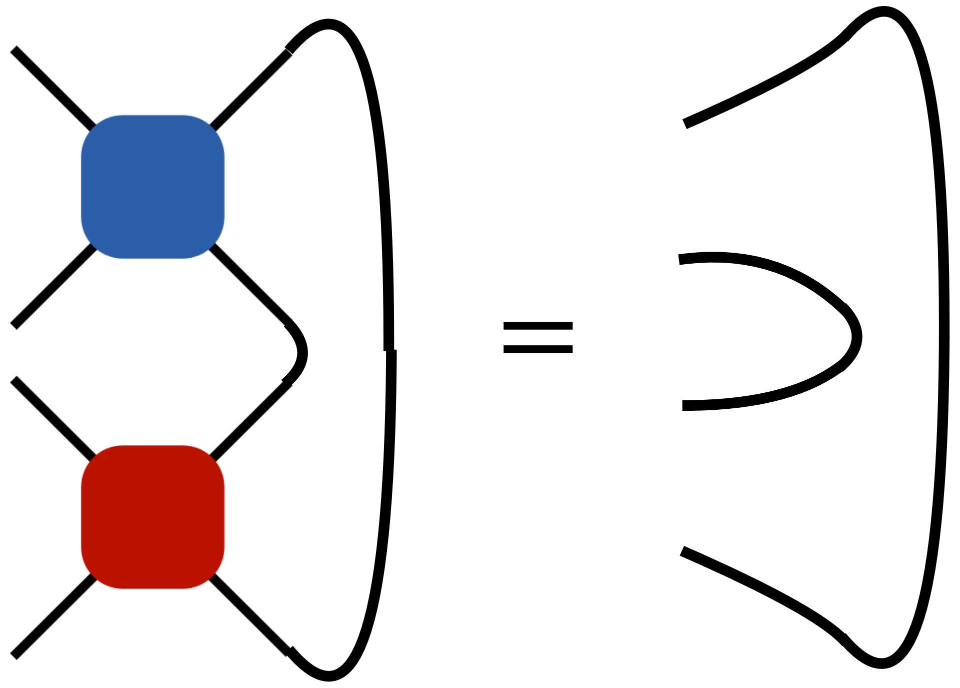
More formally, use unitarity and the diagrammatic tensor notation: U :  $U^\dagger$: 

$C_n(t) \propto$  $= 0$

dual unitarity:

$\tilde{U}\tilde{U}^\dagger$: 

$\tilde{U}^\dagger\tilde{U}$: 

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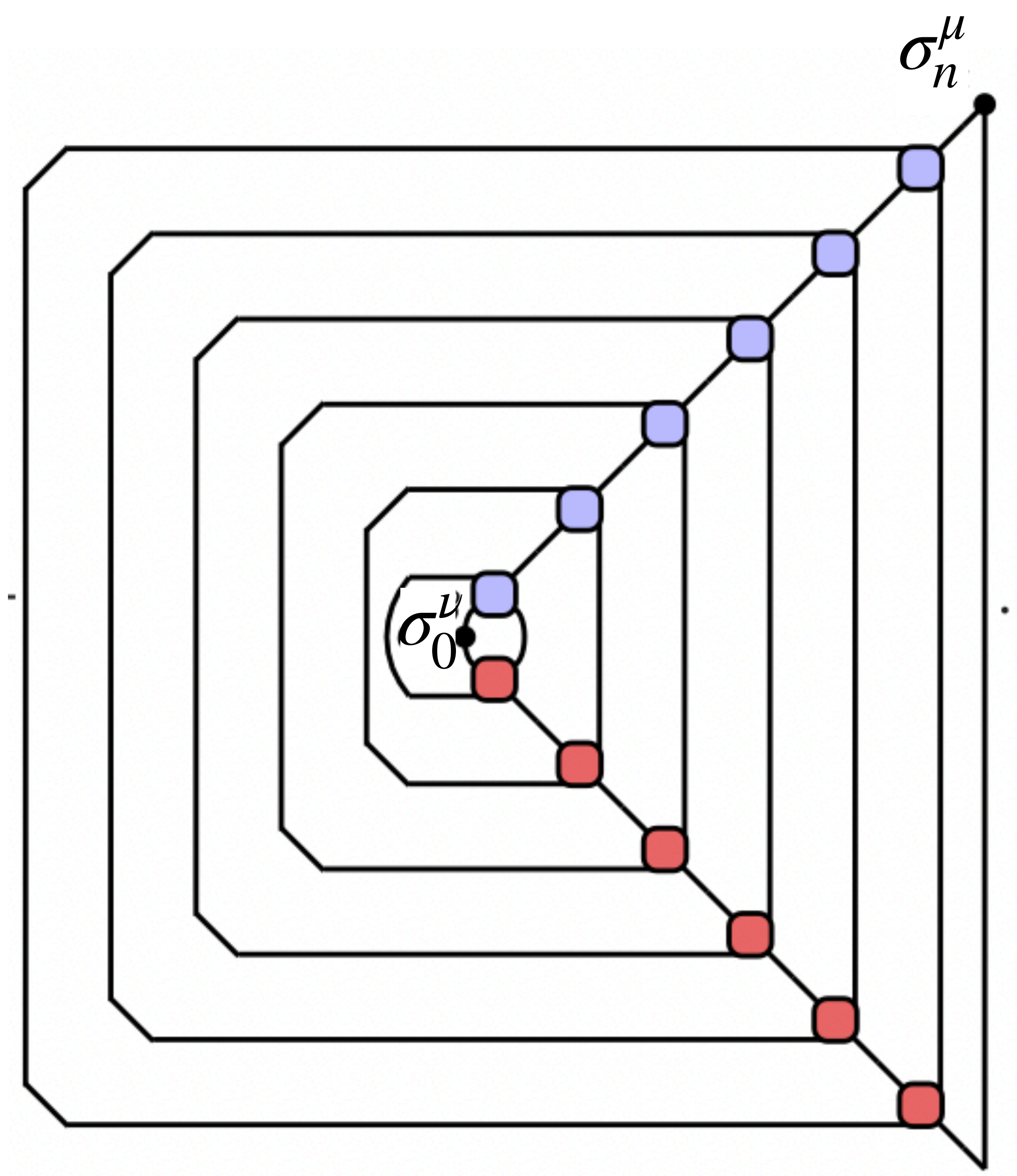
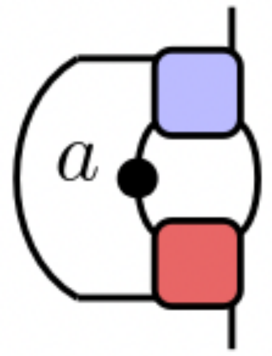
$C_t(t) \propto$

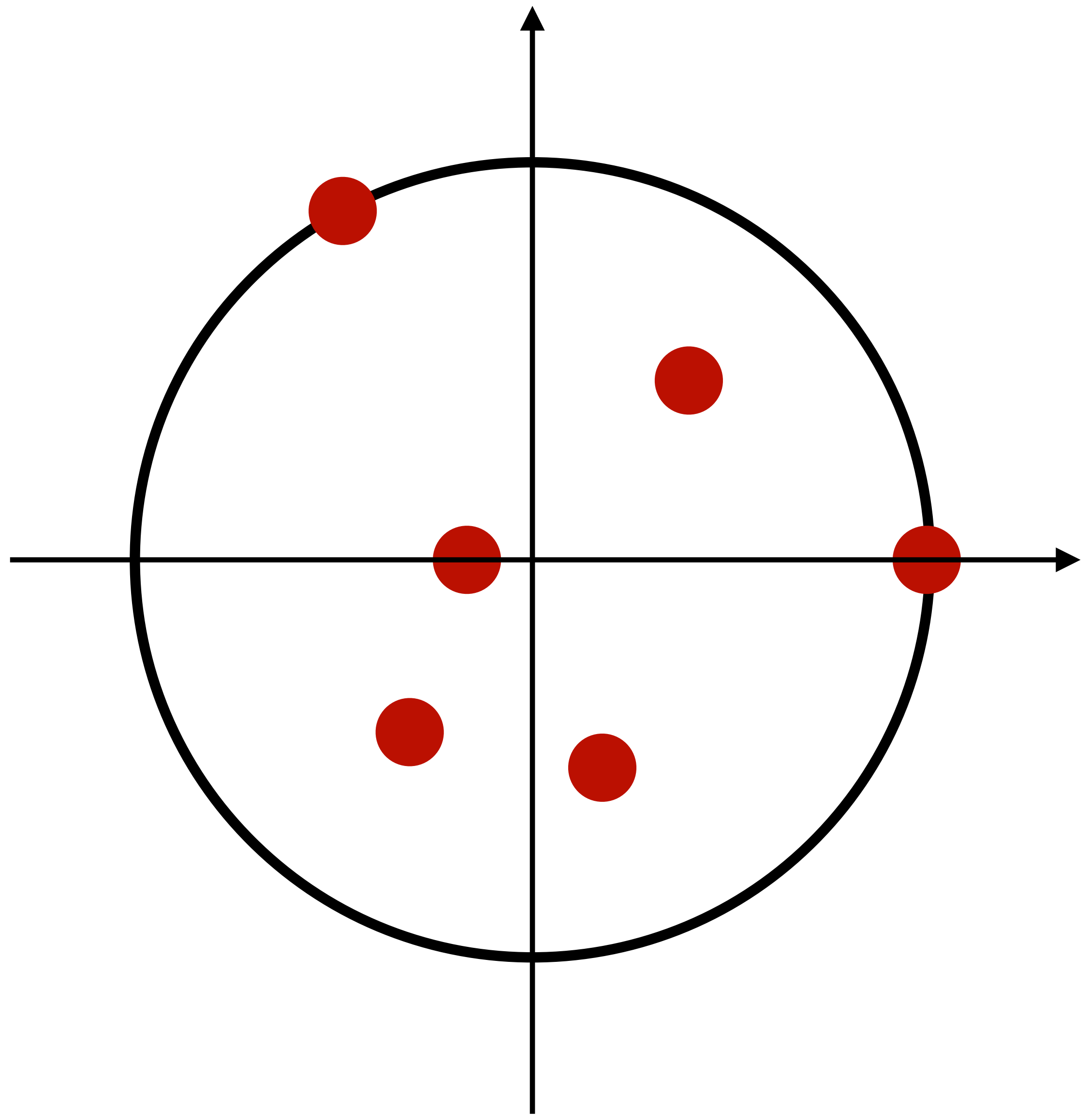


$\propto \mathrm{Tr}[\mathscr{M}^{2t}(\sigma_0^{\nu})\sigma_n^{\mu}]$

$\mathscr{M}(a) =$



eigenvalues of $\mathscr{M}$ inside unit circle



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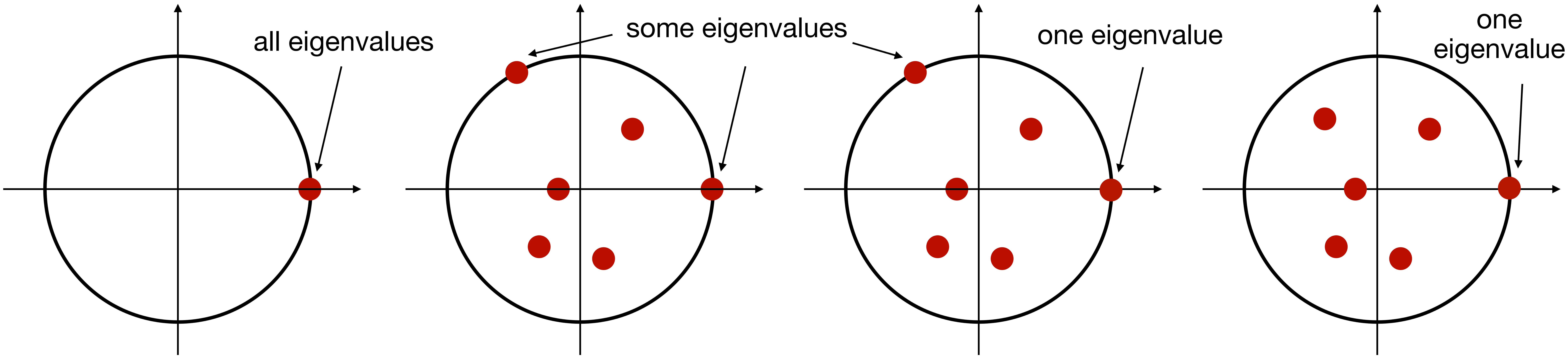
Ergodic classes:

(i) Non-interacting | (ii) Non-ergodic | (iii) Ergodic & non-mixing | (iv) Ergodic & mixing



(All correlations constant) | (Some correlations constant) | (Some correlations oscillating around equilibrium) | (All correlations decay to equilibrium)

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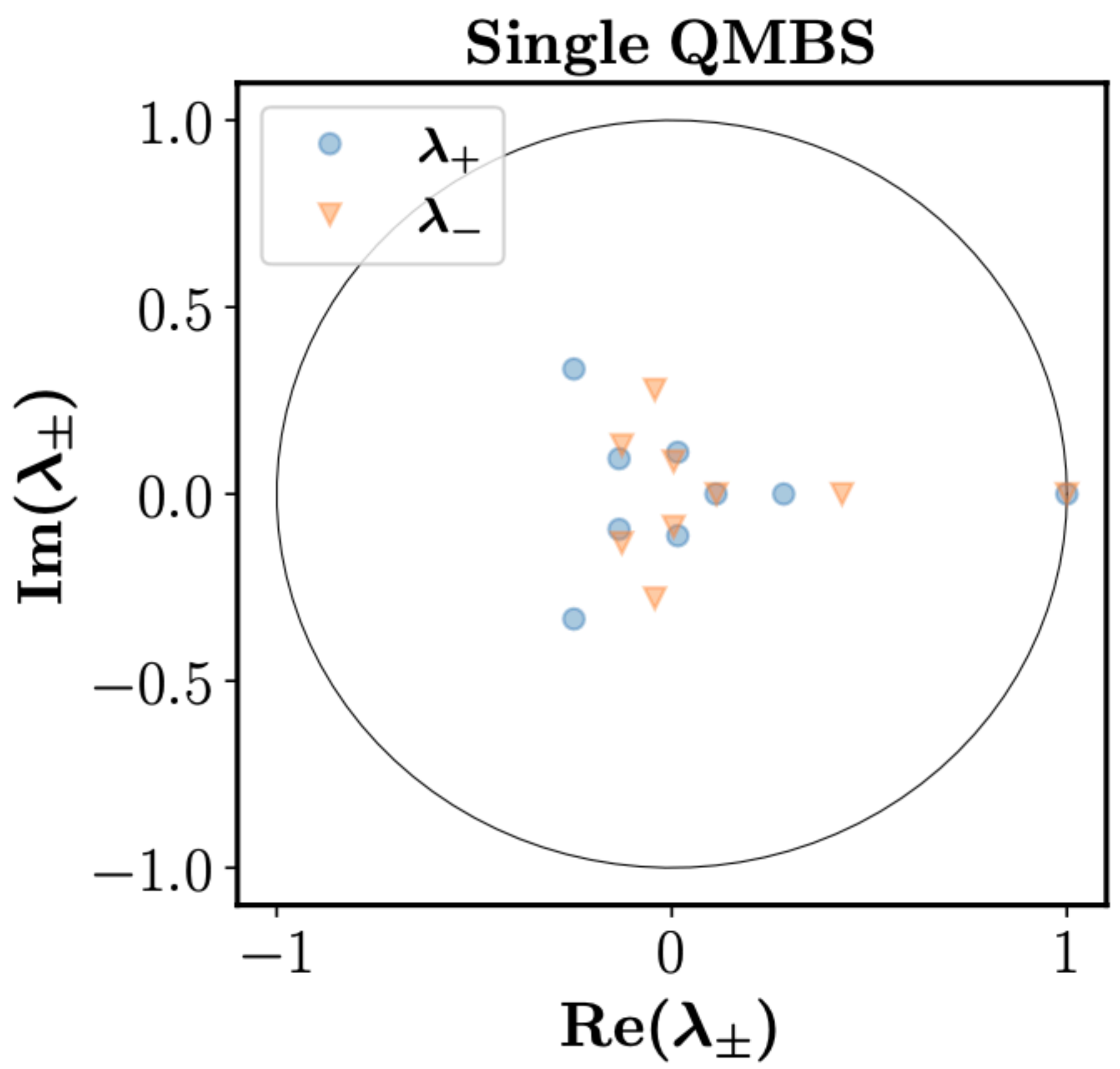
- Check ergodic properties (via eigenvalues $\lambda_{\pm}$ of the map $\mathscr{M}_{\pm}$)
  - The circuit is *ergodic* and *mixing.* ⟶

    ($C_n(t) = \frac{1}{D}\mathrm{Tr}[\hat{\mathbb{U}}^{-t}\hat{a}_n\hat{\mathbb{U}}\hat{b}_0]$ decaying exponentially except for $\hat{a}_n \propto \hat{\mathbb{I}}$)



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- By construction, circuit is DU and $\hat{\mathbb{U}}|\psi\rangle = \hat{\mathbb{S}}|\psi\rangle \in \mathscr{T}$ for $|\psi\rangle \in \mathscr{T}$.
- Check ergodic properties (via eigenvalues $\lambda_{\pm}$ of the map $\mathscr{M}_{\pm}$)
    - The circuit is *ergodic* and *mixing.*