BCS-BEC crossover in nuclear matter and related systems

Armen Sedrakian

Frankfurt Institute for Advanced Studies, Frankfurt Main
Institute of Theoretical Physics, Wrocław University

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Population imbalance

- Conventional BCS pairs particles on a Fermi surface with opposite momenta and spins in the case of *S*-wave pairing.
- There are many systems with population imbalance, where the pairing occurs between particles lying on different Fermi surfaces:

A wide class of systems, with characteristic energy scales differing by some 20 orders of magnitude, share a common feature of pairing among imbalance populations.

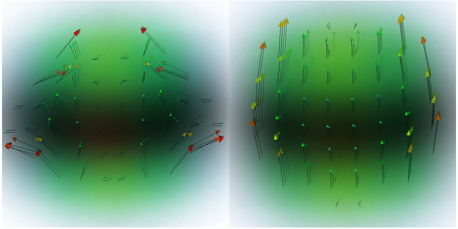
- Metallic superconductors with paramagnetic impurities. The effect of impurities is to induce an average slitting of Fermi-levels of spin-up and spin-down electrons. This can be described by adding a Pauli paramagnetic term to the spectrum:

$$\epsilon_\uparrow = \frac{p^2}{2m} - \mu_\uparrow, \quad \epsilon_\downarrow = \frac{p^2}{2m} - \mu_\downarrow, \quad \mu_\uparrow = \mu + \delta\mu, \quad \mu_\downarrow = \mu - \delta\mu, \quad \delta\mu \propto \sigma B$$

  -Concepts of "gapless superconductivity" (1961)
  -Concepts of moving condensate - "Fulde-Ferrel-Larkin-Ovchninnikov- phase" (1964)
- Nuclear system - neutron-proton pairing in nuclei and astrophysical objects
- Deconfined quark matter - pairing among different flavor of quarks

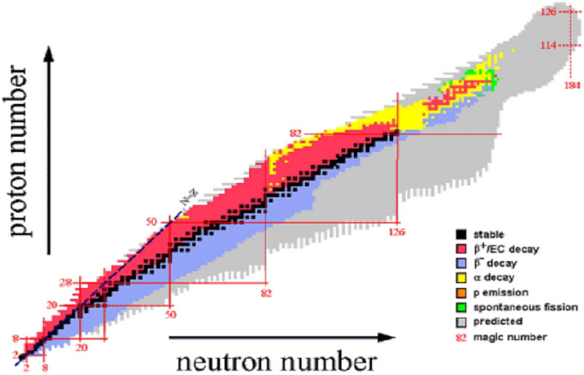
Spin polarized neutron matter in magnetars

Most of the compact stars feature field $B \sim 10^{12}$ G. But a special class of these - magentars - may feature fields of the order 10^{15} G at the surface and up to 10^{18} G in the interiors.



Effects on the strong magnetic field on the Ne nucleus via spin-paramagnetic interaction with the B field (a) $B = 0$, (b) $B = 10^{17}$ G [taken from Phys. Rev. C 94, 035802 (2016).] Computed via the Sky3D code modified to account for B-fields.

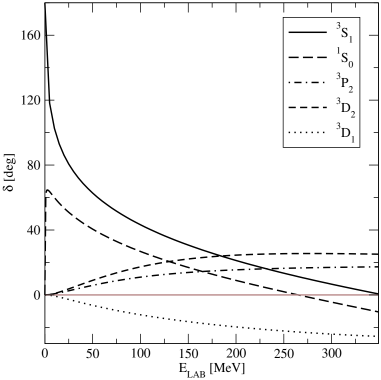
Isospin asymmetrical nuclear matter

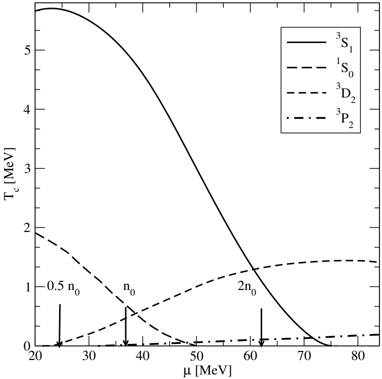


Nuclei away from valley of beta-stability are isospin asymmetrical - perhaps some traces of *n-p* pairing can be seen.

In cases when pairing is between neutrons and protons the isospin asymmetry will lead to “imbalanced pairing” !

Critical temperatures in nuclear matter





Left panel. Dependence of the experimental scattering phase shifts in the $^3S_1$, $^3P_2$, $^3D_2$, and $^3D_1$ partial waves on the laboratory energy. *Right panel.* The dependence of the critical temperatures of superfluid phase transitions in the attractive channels on the chemical potential.

Pairing in isospin asymmetrical nuclear matter

Nuclear matter Hamiltonian - isospin asymmetry - Greek indices label isospin

$$\hat{H} = \sum_{\alpha} \int d^3x \frac{1}{2m_\alpha} \nabla \hat{\psi}^\dagger_\alpha(\boldsymbol{r}) \nabla \hat{\psi}_\alpha(\boldsymbol{r}) - \sum_{\alpha\beta} \int d^3x d^3x' \hat{\psi}^\dagger_\alpha(\boldsymbol{r}) \hat{\psi}^\dagger_\beta(\boldsymbol{r}) \underbrace{V(\boldsymbol{r},\boldsymbol{r}')}_{NN-interaction} \hat{\psi}_\beta(\boldsymbol{r}') \hat{\psi}_\alpha(\boldsymbol{r}).$$

We use real time Green's function, equilibrium limit of Keldysh-Schwinger formalism.
Dyson equations (DE) for neutrons and protons

$$\hat{G}^{-1}_\alpha(x_1) \hat{G}_{\alpha\beta}(x_1,x_2) = \hat{\mathbf{1}} \delta_{\alpha\beta} \delta(x_1 - x_2) + i \sum_\gamma \int d^3x_3 \, \hat{\Sigma}_{\alpha\gamma}(x_1,x_3) \hat{G}_{\gamma\beta}(x_3,x_2),$$

where $\hat{\mathbf{1}}$ is a unit matrix, $G^{-1}_\alpha(x) \equiv i\partial/\partial t + \boldsymbol{\nabla}^2/2m_\alpha + \mu_\alpha$ and GF of the superfluid state (Nambu-Gorkov space)

$$i\hat{G}_{\alpha\beta}(x_1,x_2) \equiv i \begin{pmatrix} G_{\alpha\beta}(x_1,x_2) & F_{\alpha\beta}(x_1,x_2) \\ F^\dagger_{\alpha\beta}(x_1,x_2) & G^\dagger_{\alpha\beta}(x_1,x_2) \end{pmatrix}, \quad \hat{G}^{-1}_\alpha(x) = \begin{pmatrix} G^{-1}_\alpha(x) & 0 \\ 0 & \left[G^{-1}_\alpha(x)\right]^* \end{pmatrix}.$$

The DSE equations are closed via the approx. for the self-energy matrix (anomalous part reads)

$$\Delta_{\alpha\beta}(x_1,x_2) = \sum_{\gamma\kappa} \int \Gamma_{\alpha\beta\gamma\kappa}(x_1,x_2;x_3,x_4) F_{\gamma\kappa}(x_3,x_4) dx_3 dx_4.$$

Integrating out fast modes in real-time Green's functions

Inhomogeneous systems - separation of CM and relative motions

$$\hat{G}(x,X) \to \hat{G}(\omega,\boldsymbol{p},\mathbf{R},T), \qquad x = x_1 - x_2, \qquad X = (x_1+x_2)/2$$

The DSE now is written as

$$\sum_\gamma \begin{pmatrix} \omega - \epsilon^+_\alpha \delta_{\alpha\gamma} & -\Delta_{\alpha\gamma} \\ -\Delta^\dagger_{\alpha\gamma} & \omega + \epsilon^-_\alpha \delta_{\alpha\gamma} \end{pmatrix} \begin{pmatrix} G_{\gamma\beta} & F_{\gamma\beta} \\ F^\dagger_{\gamma\beta} & G^\dagger_{\gamma\beta} \end{pmatrix} = \delta_{\alpha\beta}\hat{\mathbf{1}}, \tag{1}$$

where *normal state spectrum*

$$\boxed{\epsilon^\pm_\alpha = \epsilon^\pm_{\alpha,\mathrm{Kin}} - \mu_\alpha \pm \mathrm{Re}\,\Sigma_\alpha - \mathrm{Im}\,\Sigma_\alpha \simeq \epsilon^\pm_{\alpha,\mathrm{Kin}} - \mu_\alpha \pm \delta\mu.} \tag{2}$$

The quasiparticle excitation spectrum is determined in the standard fashion by finding the poles of the propagators

$$\epsilon^\pm_{\alpha,\mathrm{Kin}} = \frac{1}{2m_\alpha}\left(\frac{\mathbf{P}}{2} \pm \boldsymbol{p}\right)^2 = \frac{P^2}{8m_\alpha} \pm \frac{Pp}{2m_\alpha}\cos\theta + \frac{p^2}{2m_\alpha} \tag{3}$$

Solutions of Dyson equation (spectrum)

$$\omega_{\pm\pm} = \epsilon_A \pm \sqrt{\epsilon_S + \frac{1}{2}\mathrm{Tr}\left(\Delta\Delta^\dagger\right) \pm \frac{1}{2}\sqrt{[\mathrm{Tr}\left(\Delta\Delta^\dagger\right)]^2 - 4\mathrm{Det}\left(\Delta\Delta^\dagger\right)}}.$$

$$\Delta \equiv \Delta_{\alpha\beta} \qquad \epsilon_S = (\epsilon^+ + \epsilon^-)/2, \qquad \epsilon_A = (\epsilon^+ - \epsilon^-)/2$$

Four-fold split spectrum:

- isospin asymmetry and finite momentum
- competition between spin-1 and spin-0 pairing

$$\Delta = \begin{pmatrix} \Delta_{\uparrow\downarrow} & \Delta_{\uparrow\uparrow} \\ \Delta_{\downarrow\downarrow} & \Delta_{\downarrow\uparrow} \end{pmatrix} \simeq \begin{pmatrix} \Delta_{\uparrow\downarrow} & 0 \\ 0 & \Delta_{\downarrow\uparrow} \end{pmatrix} \tag{4}$$

In nuclear matter $S-D$ isospinglet state dominates (but not in neutron stars). The spectrum under this approximation is two-fold split - complete analogue to other imbalanced systems. Solve coupled equations for densities

$$\rho_{n/p}(\vec{Q}) = -2\int\frac{d^4k}{(2\pi)^4}\mathrm{Im}[G^+_{n/p}(k,\vec{Q}) - G^-_{n/p}(k,\vec{Q})]f(\omega), \tag{5}$$

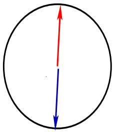
and pairing gap

$$\Delta(Q) = \frac{1}{2}\sum_{a,r}\int\frac{d^3k'}{(2\pi)^3}V_{l,l'}(k,k')\frac{\Delta_{l'}(k',Q)}{2\sqrt{E_S(k')^2+\Delta_{l'}(k',Q)}}[1-2f(E^r_a)], \tag{6}$$

where $V_{l,l'}(k,k')$ is the interaction in the $S-D$ partial wave.

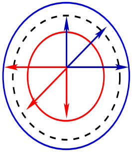
Realizations of superconducting phases with two species

BCS: $k = -k, \delta\mu = 0$



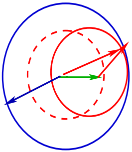
rotational/transl. symmety

ASYMMETRIC BCS: $k = -k, \quad \delta\mu \neq 0$



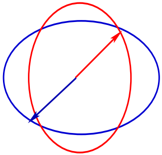
rotational/symmetry, time reversal broken

LOFF: $k + P = -k', \quad \delta\mu \neq 0$



rotational/trans sym. broken

DFS phase: $k \sim k', \quad \delta\mu \neq 0$



only rotational symmetry is broken to O(2)

Possible phases, include BCS, FFLO, DFS, and “spatial mixing” for s and n phases by a factor $0 \leq x \leq 1$

Mixed phase has domains of symmetrical BCS matter embedded in extra fluid of excess particles (no surface energy in computations yet)

$$\begin{cases} Q = 0, & \Delta \neq 0, \quad x = 0, \quad \text{BCS phase}, \\ Q \neq 0, & \Delta \neq 0, \quad x = 0, \quad \text{LOFF phase}, \\ \delta\epsilon \neq 0, & \Delta \neq 0, \quad x = 0, \quad \text{DFS phase}, \\ Q = 0, & \Delta \neq 0, \quad x \neq 0, \quad \text{PS phase}, \\ Q = 0, & \Delta = 0, \quad x = 1, \quad \text{unpaired phase}, \end{cases}$$

Thus superfluid isospin asymmetrical nuclear matter is expected to have a rich phase diagram - at least 4 competing phases - a number of critical points

BCS-BEC crossover with imbalance Occupation numbers

- Nozières-Schmitt-Rink conjecture:

  ... *the BCS theory smoothly interpolates between the weak and strong couplings.*

- Mathematically BEC limit the pair-wave function

$$\psi(k) = \langle a^{\dagger}_{n,\vec{k}} a^{\dagger}_{p,-\vec{k}} \rangle = \frac{\Delta(k)}{2E_k} \left[1 - f(E_k^{+}) - f(E_k^{-}) \right], \tag{7}$$

can be written as a Schrödinger equation

$$\frac{k^2}{m}\psi(k) + \left[1 - f(E_k^{+}) - f(E_k^{-}) \right] \sum_{k'} V(k,k')\psi_{l'}(k') = 2\mu\,\psi(k)$$

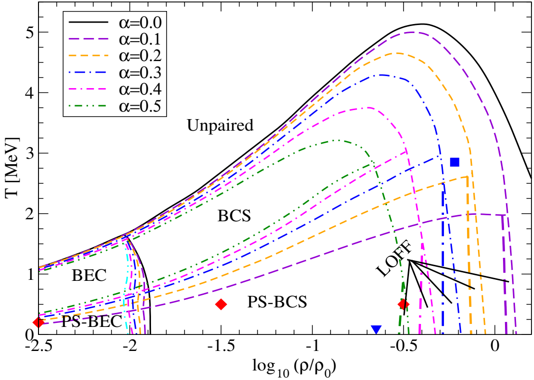
with an energy eigenvalue 2μ.

- Nuclear systems -*Density induced BCS-BEC crossover* - :

  Transition from $^3S_1$-$^3D_1$ pairing to Bose-Einstein Condensate (BEC) of deuterons

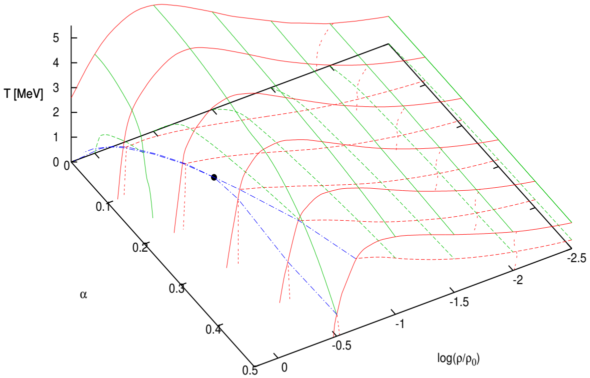
- How the BCS-BEC crossover is affected by new phases and visa-versa ?
- Does the transition remains a smooth-crossover ?
- properties of the deuteron condensate in the low-density limit ?

Temperature-density phase diagram for varying asymmetry



- Competing phases: BCS, LOFF, PS, Unpaired
- BCS - BEC crossover, with LOFF disappearing in the low density limit
- tetra-critical points (Lifshitz point), i.e., an inhomogeneous phase terminates at the point
- triangle: LOFF quenched by BCS-BEC crossover, quadrangle: quatro-critical-point

Three-dimensional view of the phase diagram



- Competing phases: BCS, LOFF, PS, Unpaired
- BCS - BEC crossover, with LOFF disappearing in the low density limit

Signatures of BCS-BEC crossover

- Pair wave function – kernel of the gap equation

$$\Psi(\boldsymbol{r}) = \sqrt{N}\int \frac{d^3p}{(2\pi)^3}[K(\boldsymbol{p},\Delta) - K(\boldsymbol{p},0)]e^{i\boldsymbol{p}\cdot\boldsymbol{r}},$$
$$K(k,\theta) \equiv \sum_{a,r} \frac{1-2f(E_r^a)}{4\sqrt{E_S(k)^2+\Delta^2(k,Q)}} \quad (8)$$

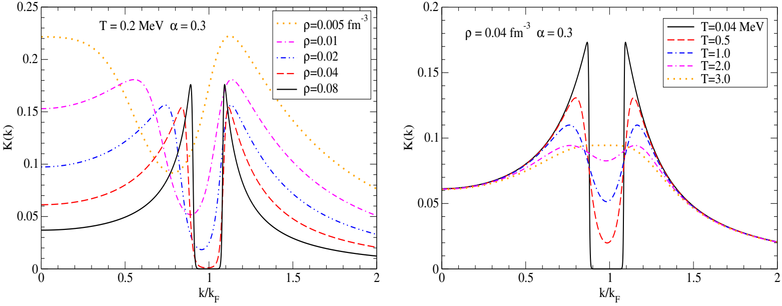
- Occupation numbers of neutrons and portons
- Quasiparticle spectra in the paired state
- Coherence length

$$\langle r^2\rangle = \int d^3r\, r^2|\Psi(\boldsymbol{r})|^2, \qquad \xi_{\rm rms} = \sqrt{\langle r^2\rangle}, \qquad \xi_a = \frac{\hbar^2 k_F}{\pi m^*\Delta}. \quad (9)$$

Consider three regimes

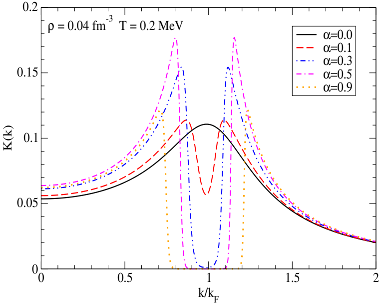
	$\log_{10}\left(\frac{\rho}{\rho_0}\right)$	$k_F[\mathrm{fm}^{-1}]$	T [MeV]	d [fm]	$\xi_{\rm rms}$ [fm]	ξ_a [fm]
WCR	−0.5	0.91	0.5	1.68	3.17	1.41
ICR	−1.5	0.42	0.5	3.61	0.94	1.25
SCR	−2.5	0.20	0.2	7.79	0.57	1.79

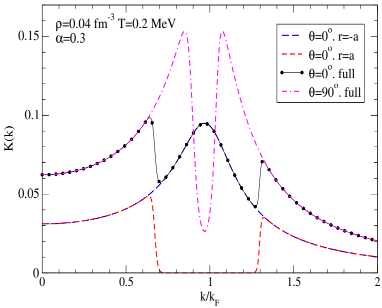
Kernel (pair-wave-function) of the gap equation in momentum space



Density and temperature dependence of the kernel of the gap (pair-wave function) equation

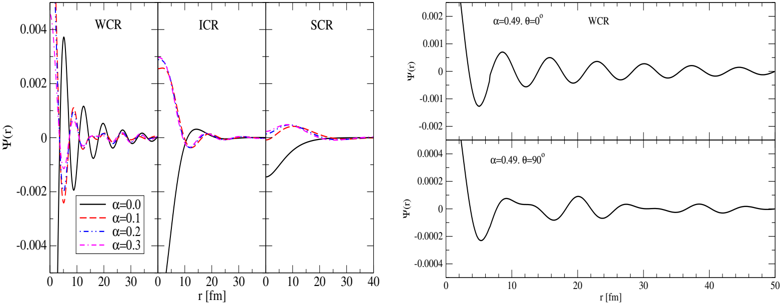
Kernel of the gap equation in momentum space





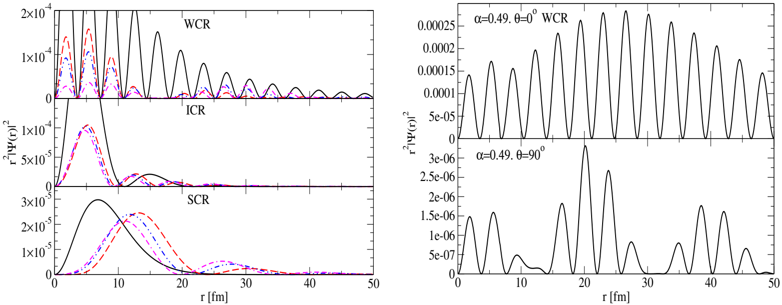
Asymmetry dependence of the kernel of the gap equation (left) and angle dependence in the case of the FFLO phase.

Real space wave function of the Cooper wave-function

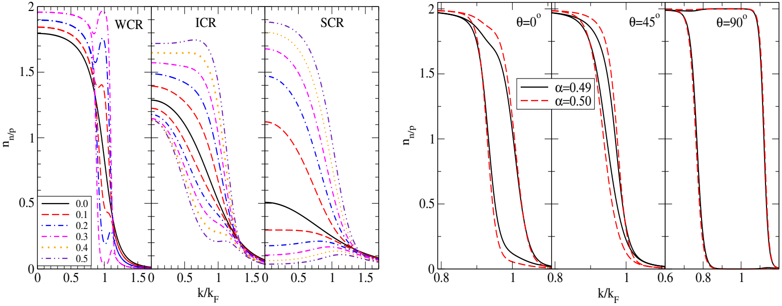


Left - asymmetry dependence across the BCS-BEC crossover; Right - angle dependence in the FFLO phase

2nd moment real space wave function of the Cooper wave-function

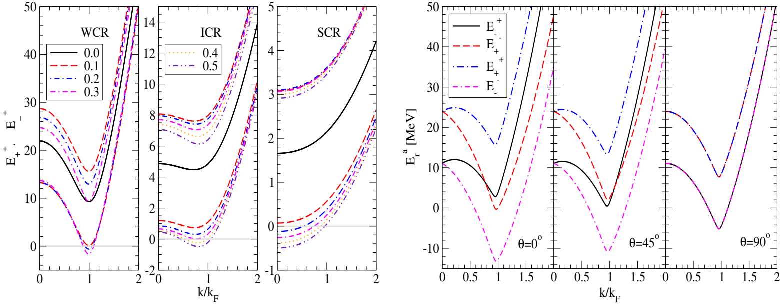


The 2nd moment of density probability, $r^2|\Psi(r)|^2$, in BCS (left) and LOFF (right) phases. Superposition of oscillations in the left lower panel is the effect of the wave-structure of the LOFF phase.



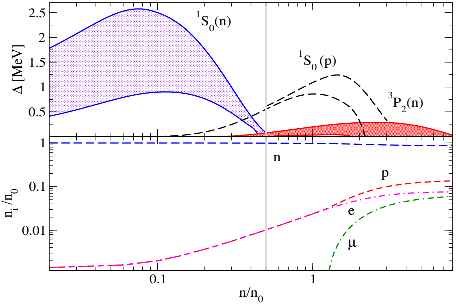
Asymmetry dependence across the BCS-BEC crossover (left). Dependence on the angle in the FFLO phase (right).

Quasiparticle spectra



Asymmetry dependence across the BCS-BEC crossover (left). Dependence on the angle in the FFLO phase (right).

Nucleonic pairing patterns in neutron stars



Three main types of condensates:

— $^1S_0$ Cooper pair condensate of neutrons in the crust
— $^1S_0$ Cooper pair condensate of protons in the core
— $^3P_2$ -$^3F_2$ Cooper pair condensate of neutron in the core
— $^3D_2$ and $^3P_0$ are attractive and can also lead to (exotic) pairing

BCS theory of spin-polarized neutron matter.

Nambu-Gorkov matrix Green's function

$$i\hat{G}_{12} = i\begin{pmatrix} G^{+}_{12} & F^{-}_{12} \\ F^{+}_{12} & G^{-}_{12} \end{pmatrix} = \begin{pmatrix} \langle T_\tau \psi_1 \psi_2^{+} \rangle & \langle T_\tau \psi_1 \psi_2 \rangle \\ \langle T_\tau \psi_1^{+} \psi_2^{+} \rangle & \langle T_\tau \psi_1^{+} \psi_2 \rangle \end{pmatrix},$$

Dyson equation, which we write in momentum space as

$$\left[\hat{G}_0(k, \boldsymbol{Q})^{-1} - \Xi(k, \boldsymbol{Q})\right] \hat{G}(k, \boldsymbol{Q}) = \mathbf{1}_{4\times4},$$

where $\Xi(k, \boldsymbol{Q})$ is the matrix self-energy. Explicitly,

$$\begin{pmatrix} ik_\nu - \epsilon^{+}_{\uparrow} & 0 & 0 & i\Delta \\ 0 & ik_\nu - \epsilon^{+}_{\downarrow} & -i\Delta & 0 \\ 0 & i\Delta & ik_\nu + \epsilon^{-}_{\uparrow} & 0 \\ -i\Delta & 0 & 0 & ik_\nu + \epsilon^{-}_{\downarrow} \end{pmatrix} \begin{pmatrix} G^{+}_{\uparrow} & 0 & 0 & F^{-}_{\uparrow\downarrow} \\ 0 & G^{+}_{\downarrow} & F^{-}_{\downarrow\uparrow} & 0 \\ 0 & F^{+}_{\uparrow\downarrow} & G^{-}_{\uparrow} & 0 \\ F^{+}_{\downarrow\uparrow} & 0 & 0 & G^{-}_{\downarrow} \end{pmatrix} = 1, \quad (10)$$

where we use short-hand $G^{+}_{\uparrow} \equiv G^{+}_{\uparrow\uparrow}$ and so on.

Quasiparticle spectra and Pauli spin-paramagnetism

These single-particle energies can be separated into symmetrical and anti-symmetrical parts with respect to time-reversal operation by writing

$$\begin{aligned}\epsilon_{\uparrow}^{\pm} &= E_S - \delta\mu \pm E_A,\\ \epsilon_{\downarrow}^{\pm} &= E_S + \delta\mu \pm E_A,\end{aligned}$$

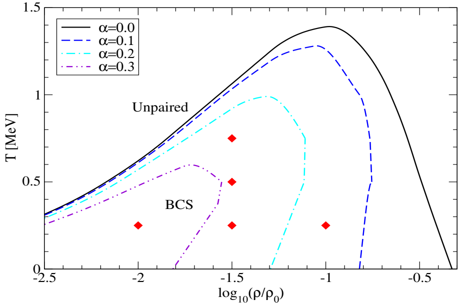
where

$$E_S = \frac{Q^2/4 + k^2}{2m^*} - \bar{\mu}, \quad E_A = \frac{\boldsymbol{k}\cdot\boldsymbol{Q}}{2m^*}, \quad \delta\mu \equiv (\mu_\uparrow - \mu_\downarrow)/2 \quad \bar{\mu} \equiv (\mu_\uparrow + \mu_\downarrow)/2.$$

The possible solutions, or phases, of the variational problem, so defined can be classified according to the alternatives

$$\begin{array}{llll} Q = 0, & \Delta \neq 0, & x = 0, & \text{BCS phase},\\ Q = 0, & \Delta = 0, & x = 1, & \text{unpaired phase},\\ Q \neq 0, & \Delta \neq 0, & x = 0, & \text{LOFF phase},\\ Q = 0, & \Delta \neq 0, & 0 < x < 1, & \text{phase-separated phase}. \end{array}$$

Phase diagram of spin-polarized neutron matter

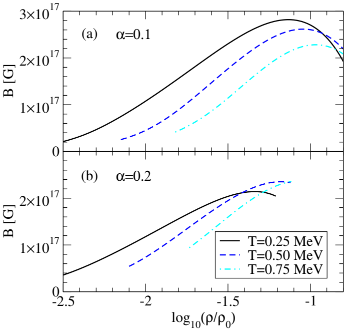


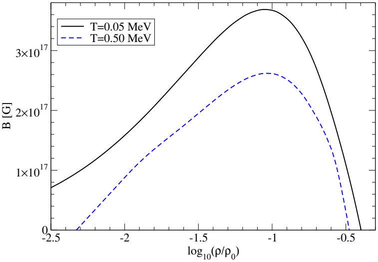
Temperature-density phase diagram of neutron matter in the temperature-density plane for several spin polarization

$$\alpha = \frac{\rho_\uparrow - \rho_\downarrow}{\rho_\uparrow + \rho_\downarrow}$$

induced by magnetic fields.

Critical magnetic field in neutron matter





– Left: Magnetic field required to create a specified spin polarization as a function of the density for two polarization values $\alpha = 0.1$ (a) and 0.2 (b) and temperatures $T = 0.25$ MeV (solid line), 0.5 MeV (dashed line), and 0.75 MeV (dash-dotted line).
– Right: Unpairing magnetic field as a function of density (in units of ρ_0) for $T = 0.05$ MeV (solid line) and $T = 0.5$ MeV (dashed line).

Conclusions

- Asymmetrical nuclear matter in the superfluid state may feature a number of unconventional phases including a phase with moving condensate. The phase diagram is complex and contains tri-critical points and even a four-critical point
- BCS-BEC crossover induced complex modification at the microscopic level: quasiparticle spectrum, occupation of states, topology of Fermi sphere, and the structure of pair wave-function.
- Spin-polarized neutron matter may feature similar phases, but now due to the spin-polarization by superstrong magnetic fields. Many parallels to the case of asymmetrical nuclear matter.
- For more information, see the review by A. S. and J. W. Clark, Eur. Phys. J. A (2019) 55: 167, arXiv:1802.00017.