Tensor network toward the lattice QCD

Shinichiro Akiyama [a), b)]

[a)] Center for Computational Sciences, University of Tsukuba

[b)] Endowed Chair for Quantum Software, The University of Tokyo

2024.9.25 @ RPMBT22

50th anniversary of lattice QCD

- **"*Confinement of quarks*" by K. G. Wilson in 1974**

Kenneth G. Wilson
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York 14850
(Received 12 June 1974)

A mechanism for total confinement of quarks, similar to that of Schwinger, is defined which requires the existence of Abelian or non-Abelian gauge fields. It is shown how to quantize a gauge field theory on a discrete lattice in Euclidean space-time, preserving exact gauge invariance and treating the gauge fields as angular variables (which makes a gauge-fixing term unnecessary). The lattice gauge theory has a computable strong-coupling limit; in this limit the binding mechanism applies and there are no free quarks. There is unfortunately no Lorentz (or Euclidean) invariance in the strong-coupling limit. The strong-coupling expansion involves sums over all quark paths and sums over all surfaces (on the lattice) joining quark paths. This structure is reminiscent of relativistic string models of hadrons.

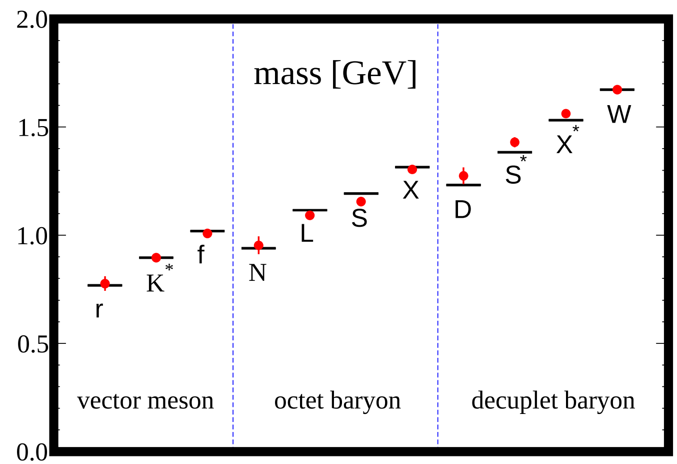
I. INTRODUCTION

The success of the quark-constituent picture

particles over short times and short distances. The polarization effects which prevent the appearance of electrons in the final state take place

Wilson, PRD10(1974)2445

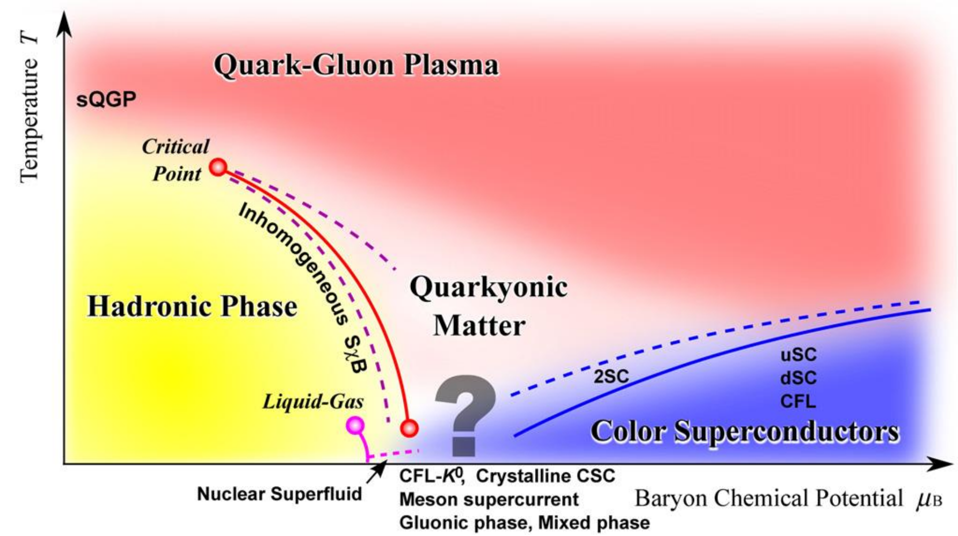
- **Successes over the past 50 years**
  - Hadron mass, matrix elements
  - Nuclear force, ...
- **Monte Carlo simulations** on supercomputers
- Great development of algorithms and supercomputers



PACS-CS Collaboration, Kuramashi+, Lattice2008

Can we solve the QCD using only the MC?

- No, unfortunately
  - MC has the **sign problem**, that has prevented numerous studies
    - **QCD at finite density**
    - Real time evolution
    - Strong CP problem
    - Chiral gauge theory
    - Other finite-density systems
    - Supersymmetry
    - ...



Fukushima-Hatsuda, Rept. Prog. Phys. 74(2011)014001

- Many attempts have been made to overcome the sign problem in the computational physics

Cf. Talk by Siu Chin on Tue, 24/9

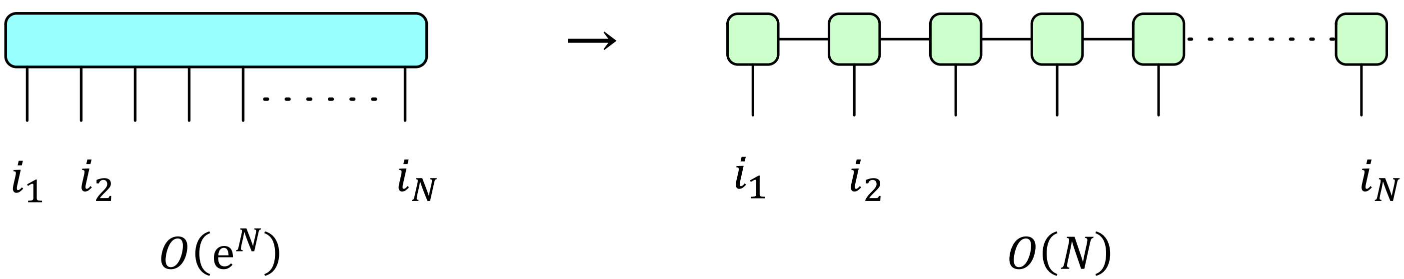
Cf. Talk by Alexander Lichtenstein on Wed, 25/9

Tensor Network?

- **Theoretical or numerical methods based on representing quantum many-body systems as a network of numerous tensors (multi-index objects)**

Cf. Talk by Garnet Kin-Lic Chan on Wed 25/9

  - Originated in statistical physics — Baxter, J. Math. Phys. 9 (1968) 650
  - Extremely successful for 1D quantum systems
    - **Density Matrix RG** — White, Phys. Rev. Lett. 69 (1992) 2863; **Feenberg Memorial Medal in 2019 (RPMBT-20)**
    - Matrix Product State (MPS) — Schollwöck, Rev. Mod. Phys. 77 (2005) 259-315



  - Accurate information compression with finite **bond dimensions** is available when the entanglement of the system is small
  - **No sign problem**

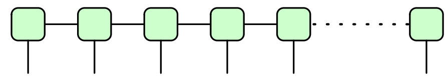
How to use TN for many-body problems

- Within the **Hamiltonian** formalism

Cf. Poster by Ryo Watanabe [Board:13]

  - TN as a variational ansatz for the **many-body state** $|\Psi\rangle$

$$|\Psi\rangle \quad =$$

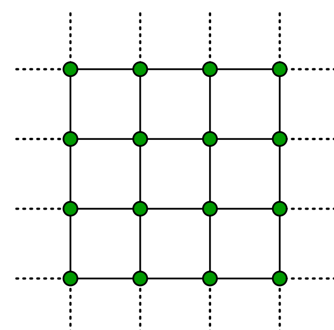


    - Determines the ground state and excited states
    - **Variational optimization** of the TN

White, Phys. Rev. Lett. 69 (1992) 2863
Nishino-Hieida-Okunishi-Maeshima-Akutsu-Gendiar, Prog. Theor. Phys. 105 (2001) 409-17
Vidal, Phys. Rev. Lett. 91 (2003) 147902
Verstraete-Cirac, arXiv:cond-mat/0407066
…

- Within the **Lagrangian** formalism
  - **Path integral** Z is described as a TN
    - Contraction of the TN
    - **Real-space RG** on the TN

$$Z \quad =$$



Nishino-Okunishi, J. Phys. Soc. Japan 65 (1996) 891
Levin-Nave, Phys. Rev. Lett. 99 (2007) 120601
Gu-Wen, Phys. Rev. B80(2009)155131
Evenbly-Vidal, Phys. Rev. Lett. 115(2015)180405
…

TN toward the lattice QCD

- "Kogut ladder" can be a good roadmap
  - Gradually try to increase symmetry and dimensionality
- Advantage of TN over MC
  - No sign problem
  - Can directly deal with **fermions**
  - Thermodynamic limit can be handled directly when the system has the translational symmetry
- What should be addressed
  - How to regularize **bosons**
  - Accuracy vs computational cost

Cf. "A quantum-simulation program for QCD?" by Zohreh Davoudi in RPMBT-21

II. LATTICE FIELD THEORY

A. The Kogut sequence: From Ising to QCD

In the early 1970s, QCD appeared to be a strong candidate for a theory of strong interactions involving quarks and gluons. However, the perturbative methods that provided satisfactory ways to handle the electroweak interactions of leptons failed to explain confinement, mass gaps, and chiral symmetry breaking. A nonperturbative definition of QCD was needed. In 1974, Wilson proposed (Wilson, 1974) a lattice formulation of QCD where the SU(3) local symmetry is exact. As this four-dimensional model is fairly difficult to handle numerically, a certain number of research groups started considering simpler lattice models in lower dimensions and then increased symmetry and dimensionality. This led to a sequence of models, sometimes called the "Kogut ladder," that appears in the reviews of Kogut (1979, 1983) and was later addressed with small modifications by Polyakov (1987) and Itzykson and Drouffe (1991).

The sequence is approximately the following:

(1) $D = 2$ Ising model
(2) $D = 3$ Ising model and its gauge dual
(3) $D = 2$ O(2) spin and Abelian Higgs models
(4) $D = 2$ fermions and the Schwinger model
(5) $D = 3$ and 4U(1) gauge theory
(6) $D = 3$ and 4 non-Abelian gauge theories
(7) $D = 4$ lattice fermions
(8) $D = 4$ QCD

Meurice-Sakai-Unmuth–Yockey, Rev. Mod. Phys. 94 (2022)025005

Starting from 2D, (1+1)D systems

- **Schwinger model (= 2D QED)**
  - The most QCD-like theory in 2D
    - Chiral symmetry breaking
    - Confinement
    - Topological term
  - The first application of TN to the lattice gauge theory
    - Critical property at $\theta = \pi$
      - 2D Ising universality

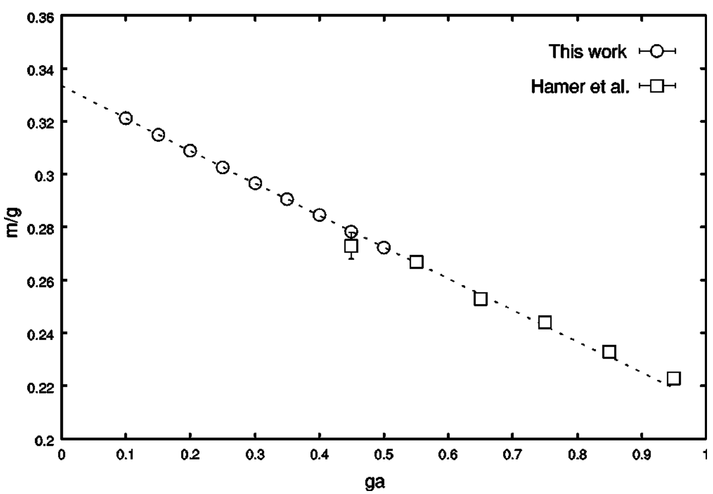
$$\left(\frac{m}{g}\right)_c = 0.3335(2).$$

$$\nu = 1.01(1)$$

$$\beta/\nu = 0.125(5).$$

Byrnes-Sriganesh-Bursill-Hamer, PRD66(2002)013002

Bañuls-Cichy-Jansen-Cirac, JHEP13(2013)158
Byrnes-Haegeman-Acoleyen-Verschelde-Verstraete, PRL113(2014)091601
Shimizu-Kuramashi, PRD90(2014)014508, 074503
Bañuls-Cichy-Cirac-Jansen-Kühn, PRL118(2017)071601
Pichler-Dalmonte-Rico-Zoller-Montangero, PRX6(2016)011023
Saito-Bañuls-Cichy-Cirac-Jansen, Lattice2014, Lattice2015
Zapp-Orús, PRD95(2017)114508
Shimizu-Kuramashi, PRD97(2018)034502
Magnifico-Vodola-Ercolessi-Kumar-Müller-Bermudez, PRD99(2019)014503
Magnifico-Vodola-Ercolessi-Kumar-Müller-Bermudez, PRB100(2019)115152
Funcke-Janse-Kühn, PRD101(2020)054507
Butt-Catterall-Meurice-Sakai-Unmuth−Yockey, PRD101(2020)094509
Honda-Itou-Tanizaki, JHEP11(2022)141
Angelides-Funcke-Janse-Küh, PRD108(2023)0145156
Dempsey-Klebanov-Benjamin-Søggard-Zan, PRL132(2024)031603
Yosprakob-Nishimura-Okunishi, JHEP11(2023)187
Itou-Matsumoto-Tanizaki, JHEP11(2023)231, arXiv:2407.11391
Kanno-SA-Murakami-Takeda, Lattice2024
...



Starting from 2D, (1+1)D systems

- **2D QCD**
  - Testing ground for non-Abelian gauge theories
    - **SU(N) DoFs have to be regularized**
      - Removing the gauge DoFs
      - Irreducible representation
      - Quantum link formulation
      - Qubit regularization
      - Subgroups, Sampling, ...
    - Regularization is also necessary in quantum computing

Kühn-Zohar-Cirac- Bañuls, JHEP07(2015)130
Silvi-Rico-Dalmonte-Tschirsich-Montangero, Quantum1(2017)9
Bañuls-Cichy-Cirac-Jansen-Kühn, PRX7(2017)041046
Sala-Shi-Kühn-Bañuls-Demler-Cirac, PRD98(2018)034505
Silvi-Sauer-Tschirsich-Montangero, PRD100(2019)074512
Bazavov-Catterall-Jha-Unmuth−Yockey, PRD99(2019)114507
Fukuma-Kadoh-Matsumoto, PTEP2021(2021)123B03
Hirasawa-Matsumoto-Nishimura-Yosprakob, JHEP12(2021)011
Rigobello-Magnifico-Silvi-Montangero, arXiv:2308.04488
Bloch-Lohmayer, NPB986(2023)116032
Liu-Bhattacharya-Chandrasekharan-Gupta, arXiv:2312.17734
Hayata-Hidaka-Nishimura, JHEP07(2024)106
Asaduzzaman-Catterall-Meurice-Sakai-Toga, JHEP05(2024)195
Samberger-Bloch-Lohmayer, Lattice2024
Ho Pai-SA-Todo, Lattice2024
...

Horn, Phys. Lett. B100(1981)149-151
Orland-Rohrlich, NPB338(1990)647-672
Chandrasekharan-Wiese, NPB492(1997)455-471
Brower- Chandrasekharan-Wiese, PRD60(1999)094502
...

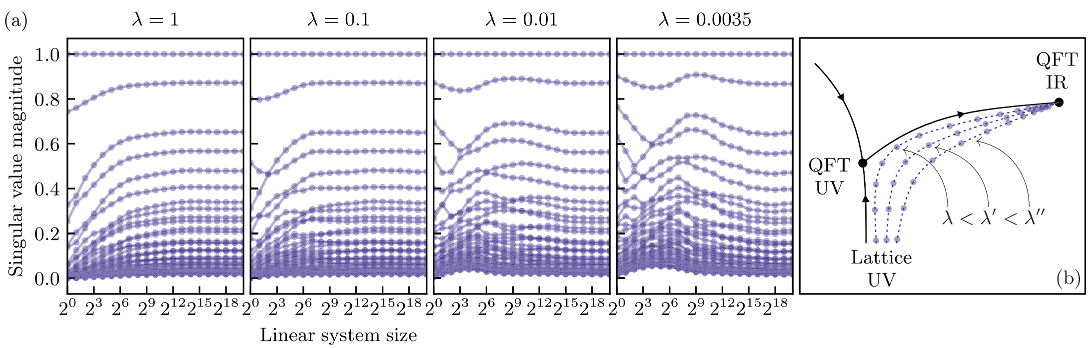
Singh-Chandrasekharan, PRD100(2019)054505
...

Cf. Bañuls+, Eur. Phys. J. D74(2020)165
...

Starting from 2D, (1+1)D systems

- **Other models:** ϕ^4 theory
  - RG prescription within TN

Sugihara, JHEP05(2004)007
Milsted-Haegeman-Osborne, PRD88(2013)085030
Gillman-Rajantie, PRD96 (2017)094509, PRD97(2018)094505
Shimizu, Mod. Phys. Lett. A27(2012)1250035
Kadoh-Kuramashi-Nakamura-Sakai-Takeda-Yoshimura, JHEP03(2018)141, JHEP05(2019)184
Zahra-Takeda-Yamazaki, Lattice2024
...



Delcamp-Tilloy, PRR2(2020)033278

- Critical coupling in the continuum limit by various numerical methods

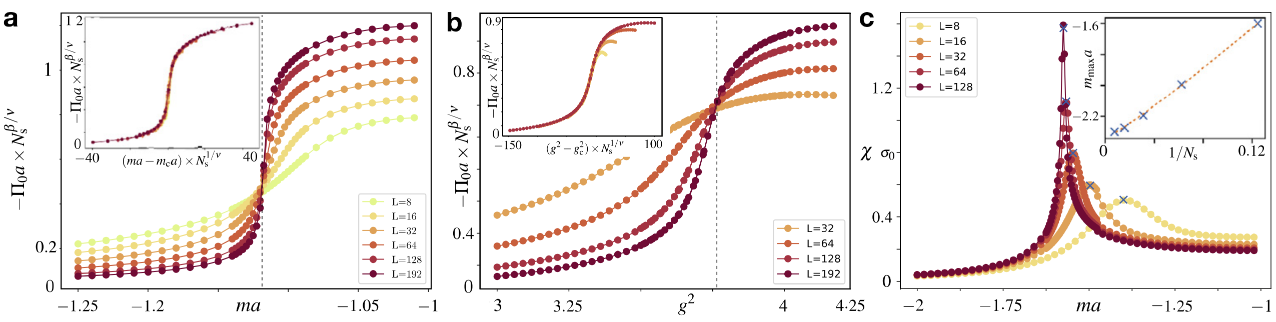
Method	Critical coupling
MPS Milsted+, PRD88(2013)085030	11.064(20)
Monte Carlo Bronzin+, PRD99(2019)034508	11.055(20)
TRG Kadoh+, JHEP05(2019)184	10.913(56)
Gilt-TNR Delcamp+, PRR2(2020)033278	11.0861(90)
MPS Vanhecke+, arXiv:2104.10564	11.09698(31)

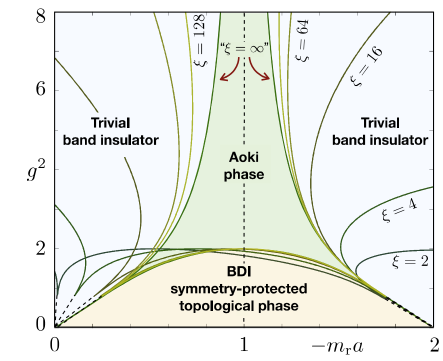
Starting from 2D, (1+1)D systems

- **Other models:** Four-fermion interactions
  - Toy models for QCD
    - Gross-Neveu model
  - Relation to strongly-correlated systems
    - Aoki phase (parity-symmetry breaking phase)
    - Topological phases

Takeda-Yoshimura, PTEP2015(2015)043B01
Jünemann-Piga-Ran-Lewenstein-Rizzi-Bermudez, PRX7(2017)031057
Bermudez-Tirrito-Rizzi-Lewenstein-Hands, Ann. Phys. NY399(2018)148
Tirrito-Rizzi-Sierra-Lewenstein-Bermudez, PRB99(2019)125106
Bañuls-Cichy-Kao-Lin-Lin-Tan, PRD100(2019)094504
SA, PRD108(2023)034514
Bañuls-Cichy-Hung-Kao-Lin-Singh, arXiv:2407.11295
...

Aoki, PRD30(1984)2653





Bermudez+, Ann. Phys. NY399(2018)148

Starting from 2D, (1+1)D systems

- And other models
  - **Zn and U(1) gauge theories**
    - Topological term
    - Toward chiral gauge theories
  - **O(N) sigma model, CP(N-1) model**
    - Study of entanglement entropy
    - Topological term
      - Haldane’s conjecture
  - **SUSY on a lattice**
  - **Euclidean quantum gravity**

Sugihara, JHEP07(2005)022
Unmuth−Yockey-Zhang-Bazavov-Meurice-Tsai, PRD98(2018)094511
Kuramashi-Yoshimura, JHEP04(2020)089
SA-Kuramashi, JHEP09(2024)086
…

Milsted, PRD93(2016)085012
Yang-Liu-Zou-Xie-Meurice, PRE93(2016)012138
Kawauchi-Takeda, PRD93(2016)114503
Bazavov-Meurice-Tsai-Unmuth−Yockey-Yang-Zhang, PRD96(2017)034514
Bruckmann-Jansen-Kühn, PRD99(2019)074501
Nakayama-Funcke-Jansen-Kao-Kühn, PRD105(2022)054507
Luo-Kuramashi, JHEP03(2024)020, arXiv:2406.08865
Aizawa-Takeda, Lattice2024
…

Kadoh-Kuramashi-Nakamura-Sakai-Takeda-Yoshimura, JHEP03(2018)141

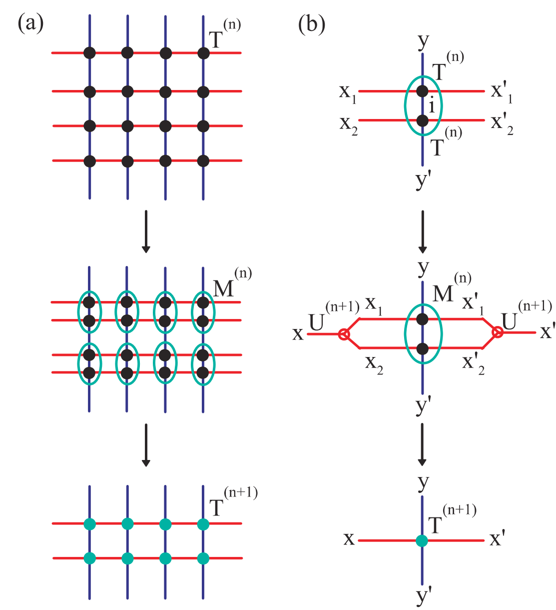
Dittrich-Mizera-Steinhaus, New Jour. Phys. 18(2016)053009
Asaduzzaman-Catterall-Unmuth−Yockey, PRD102(2020)054510
Ito-Kadoh-Sato, PRD106(2022)106044

Now, moving on to the higher dimensions

- TN computations in higher dimensions (3D, 4D) are challenging
  - Gauge DoFs cannot be eliminated as in (1+1)D
  - “Curse of dimensionality”
  - As dimensions increase, so do computational costs
    - Computational memory
    - Execution time
  - It seems difficult to extend a method that is efficient in (1+1)D
  - Development of the algorithms specialized for higher dimensions is of essential importance
  - **Several promising candidates**

Higher-dimensional TN algorithms

- **Tensor Renormalization Group (TRG)**
  - Real-space RG on tensor networks, whose accuracy can be systematically improved by increasing the bond dimension χ
    - **Higher-Order TRG (HOTRG)** Xie-Chen-Qin-Zhu-Yang-Xiang, PRB86(2012)045139
      - Computationally demanding: $O(\chi^{4d-1})$
    - **Anisotropic TRG (ATRG)** Adachi-Okubo-Todo, PRB102(2020)054432
      - Less demanding than HOTRG: $O(\chi^{2d+1})$
    - **Triad TRG** Kadoh-Nakayama, arXiv:1912.02414
      - Based on three-leg tensors: $O(\chi^{d+3})$
    - **Minimally-Decomposed TRG** Nakayama, arXiv:2307.14191
      - Comparable accuracy to HOTRG: $O(\chi^{2d+1})$



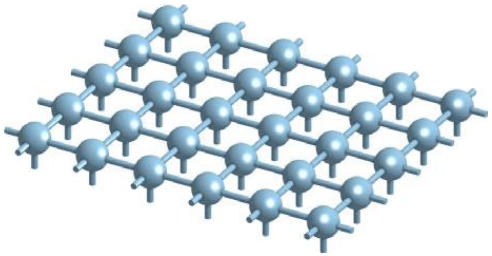
Xie+, PRB86(2012)045139

Higher-dimensional TN algorithms

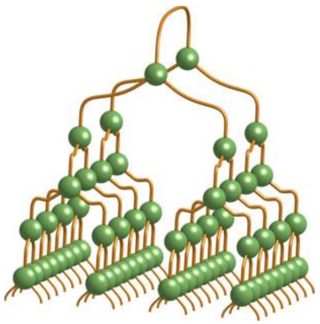
- **Tensor Network State (TNS)**
  - Variational wave function
    - **(infinite) Projected Entangled Pair States (PEPS)**

      Verstraete-Cirac, arXiv:cond-mat/0407066
      Cf. "Tensor Product State" by Nishino+, Prog. Theor. Phys. 105 (2001) 409-17
      Corboz-Orús-Bauer-Vidal, PRB81(2010)165104
      …
      - The natural generalization of MPS
      - Satisfies the area law of entanglement entropy
    - **Tree Tensor Network (TTN)**

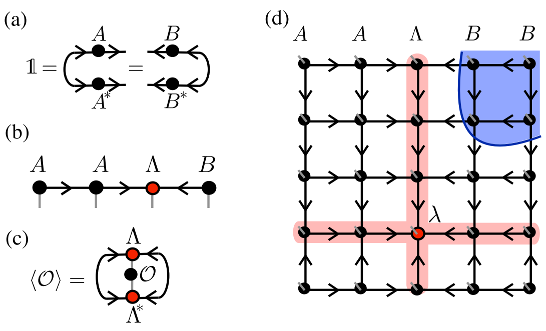
      Shi-Duan-Vidal, PRA74(2006)022320
      Tagliacozzo-Evenbly-Vidal, PRB80(2009)235127
      Silvi-Giovannetti-Montangero-Rizzi-Cirac-Fazio, PRA81(2010)062335
      …
      - Hierarchical tensor network w/o loops
      - Similarity with the real-space RG
    - **Isometric TNS (isoTNS)** Zaletel-Pollmann, PRL124(2020)037201
      - Generalization of the isometric condition of MPS to higher dimensions



Felser+, PRX10(2020)041040



Felser+, PRX10(2020)041040

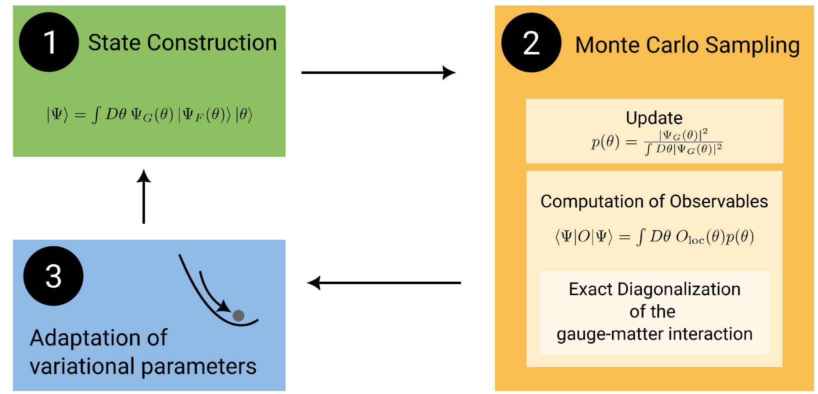


Zaletel+, PRL124(2020)037201

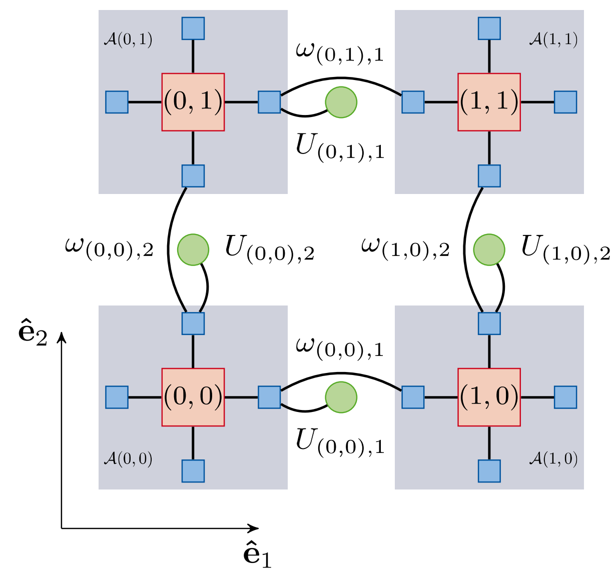
Higher-dimensional TN algorithms

- **Gauged Gaussian PEPS (GGPEPS)**
  - **Deals with the gauge DoFs with the MC sampling rather than discretizing them**
    - Constructs an ansatz state as PEPS
    - The probability is described by the norms of states
    - **Sign-problem-free MC simulation**

Zohar-Burrello, New J. Phys.18(2016)043008
Zohar-Cirac, PRD97(2018)034510
...
[Review] Kelman-Borla-Gomelski-Elyovich-Roose-Emonts-Zohar, arXiv:2404.13123



Bender-Emonts-Cirac, PRR5(2023)043128



Emonts- Bañuls-Cirac-Zohar, PRD102(2020)074501

Status of TN in 3D, (2+1)D

- Ising, Potts models [TRG, TNS]

Xie-Chen-Qin-Zhu-Yang-Xiang, PRB86(2012)045139
Dai-Cho-Batchelor-Zhou, PRE89(2014)062142
Wang-Xie-Cheng-Bruce-Xiang, Chin.Phys.Lett.31(2014)070503
Bloch-Lohmayer-Schweiß-Unmuth−Yockey, Lattice2021
Jha, arXiv:2201.01789
Kadow-Pollmann-Knap, PRB107(2023)205106

- ϕ^4 theory [TRG]

SA-Kuramashi-Yoshimura, PRD104(2021)034507

- O(N) models [TRG]

Bloch-Jha-Lohmayer-Meister, PRD104(2021)094517
SA-Jha-Unmuth−Yockey, PRD110(2024)034519

- Zn gauge theories [TRG, GGPEPS, TNS]

Tagliacozzo-Vidal, PRB83(2011)115127
Tagliacozzo-Celi-Lewenstein, PRX4(2014)041024
Kuramashi-Yoshimura, JHEP08(2019)023
Emonts- Bañuls-Cirac-Zohar, PRD102(2020)074501
Crone-Corboz, PRB101(2020)115143
Robina-Bañuls-Cirac, PRL126(2021)050401
SA-Kuramashi, JHEP05(2022)102
Emonts-Kelman-Borla-Moroz-Gazit-Zohar, PRD107(2023)014505
Nakayama-Schneider, arXiv:2407.14226

- Infinite-coupling U(N) model [TRG]

Milde-Bloch-Lohmayer, Lattice2021

- QED [TNS, GGPEPS]

Zohar-Burrello-Wahl-Cirac, Ann. Phys. 363(2015)84
Zapp-Orús, PRD95(2017)114508
Felser-Silvi-Collura-Montangero, PRX10(2020)041040
Bender-Emonts-Cirac, PRR5(2023)043128

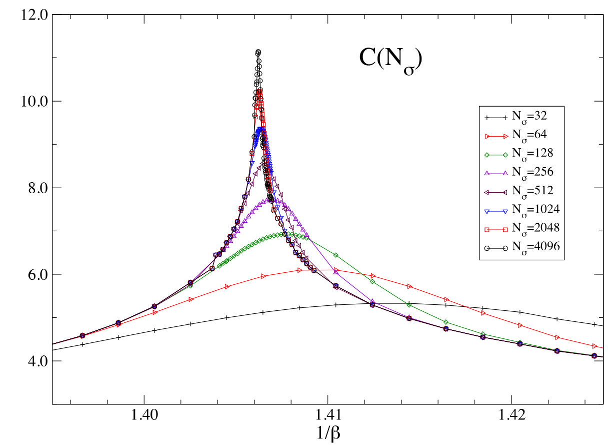
- SU(N) gauge theories [TRG, TNS]

Zohar-Wahl-Burrello-Cirac, Ann. Phys. 374(2016)84
Cataldi-Magnifico-Silvi-Montangero, PRR6(2024)033057
Kuwahara-Tsuchiya, PTEP022(2022)093B02
Yosprakob-Okunishi, arXiv:2406.16763

Z_2 gauge theories w/ TRG

- Second-order phase transition at finite temperature
  - TRG confirmed the 2D Ising universality as conjectured by Svetitsky and Yaffe

    Svetitsky-Yaffe, NPB210(1982)423



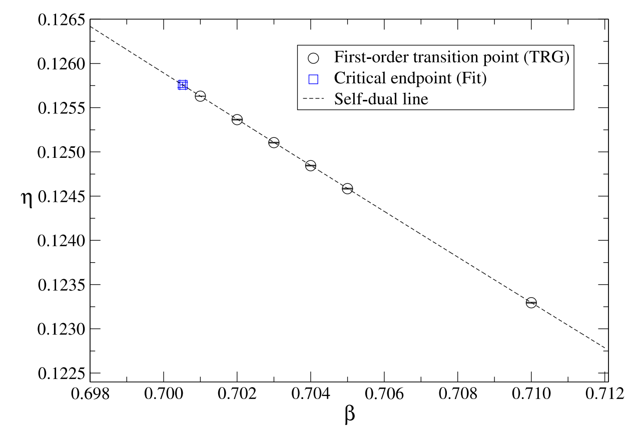
Kuramashi+, JHEP08(2019)023

- Phase transitions in 3D Z_2 gauge-Higgs model
  - TRG confirmed the self-dual transition line

    Balian-Drouffe-Itzykson, PRD11(1975)2098

  - Resulting critical endpoint is consistent with the latest MC estimation

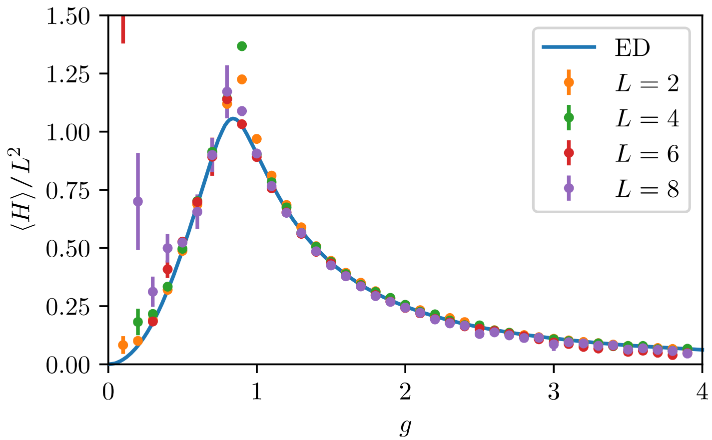
Method	Critical endpoint
MC Somoza+, PRX11(2021)041008	0.701
TRG SA+, JHEP05(2022)102	0.70051(7)

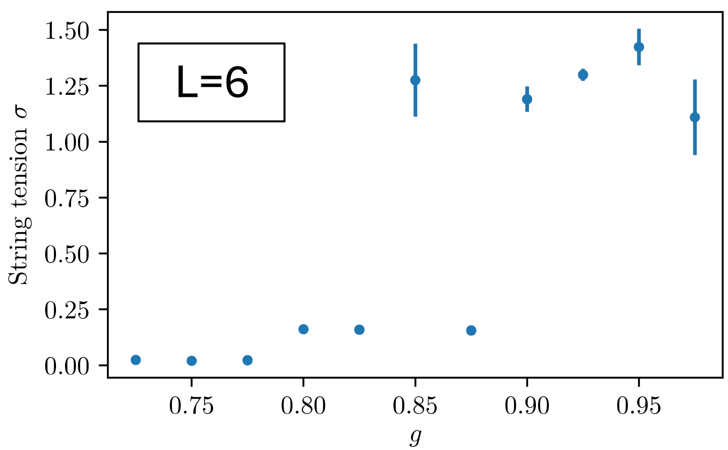


SA+, JHEP05(2022)102

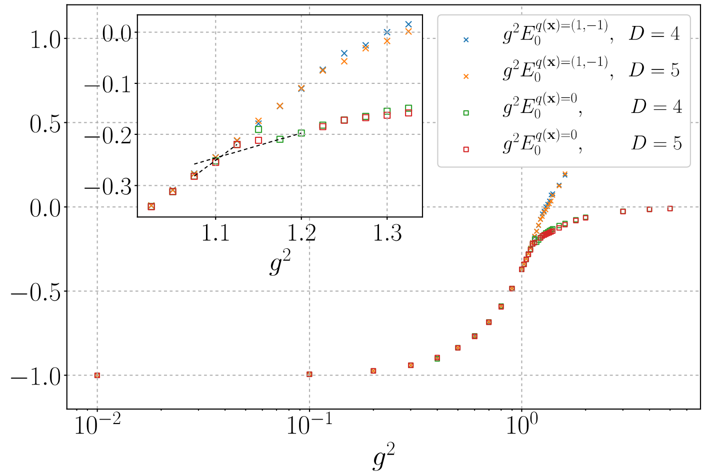
Z_3 gauge theory w/ GGPEPS, iPEPS

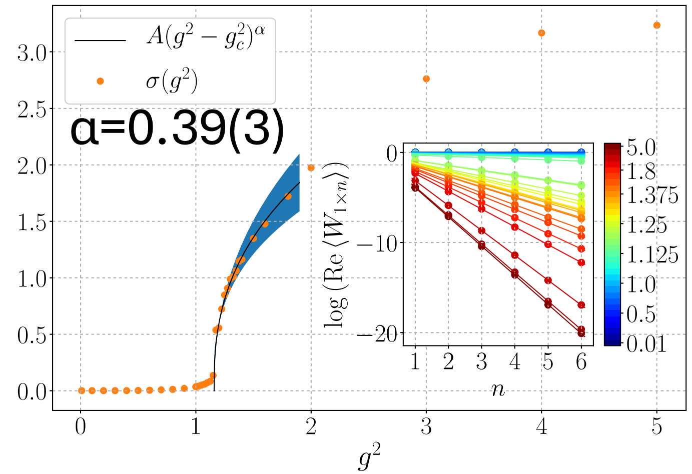
- Ground state energy and string tension by GGPEPS Emonts+, PRD102(2020)074501





- Ground state energy and string tension by iPEPS Robina+, PRL126(2021)050401



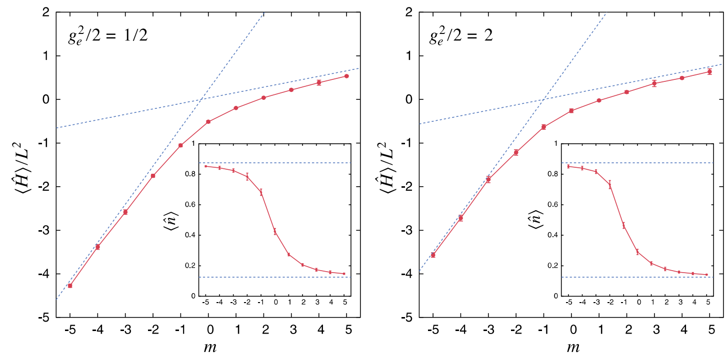


g^2_c=1.159(4)

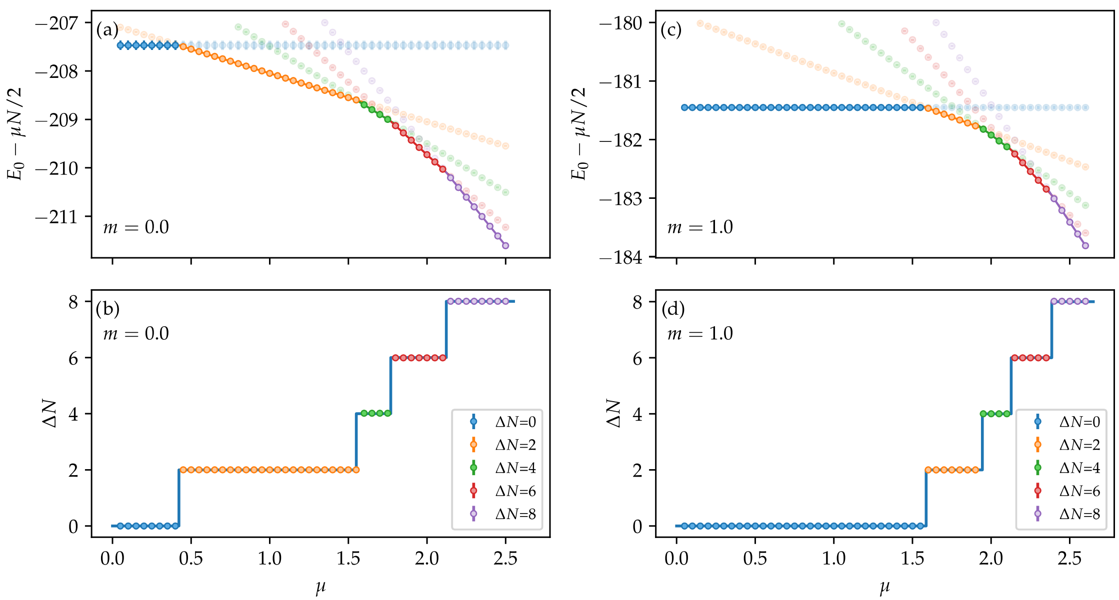
QED w/ TTN, GGPEPS

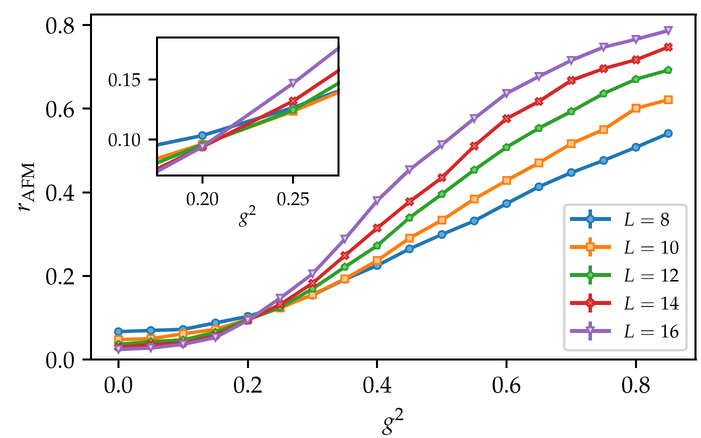
- TTN simulation w/ finite magnetic coupling
  - Ground state energy on an L=8 lattice is compared with the perturbation theory
- Confinement-deconfinement phase transition is captured by GGPEPS
- Finite-density computations are also done by GGPEPS, avoiding the sign problem

Felser+, PRX10(2020)041040



Method	Transition point
MC Xu+, PRX9(2019)021022	0.40(5)
GGPEPS Bender+, PRR5(2023)043128	0.15(2)





Bender+, PRR5(2023)043128

Status of TN in 4D, (3+1)D

- Ising model [TRG]

  SA-Kuramashi-Yamashita-Yoshimura, PRD100(2019)054510
  Sugimoto-Sasaki, Lattice2024

- ϕ^4 theory [TRG]

  SA-Kadoh-Kuramashi-Yamashita-Yoshimura, JHEP09(2020)177
  SA-Kuramashi-Yoshimura, PRD104(2021)034507

- Zn gauge-Higgs model at finite density [TRG]

  SA-Kuramashi, JHEP05(2022)102, 10(2023)077

- Infinite-coupling U(N) model [TRG]

  Milde-Bloch-Lohmayer, Lattice2021

- Nambu-Jona−Lasinio model at finite density [TRG]

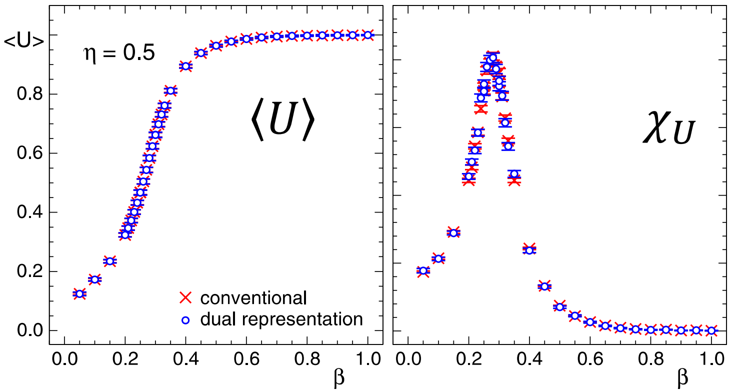
  SA-Kuramashi-Yamashita-Yoshimura, JHEP01(2021)121

- QED [TNS]

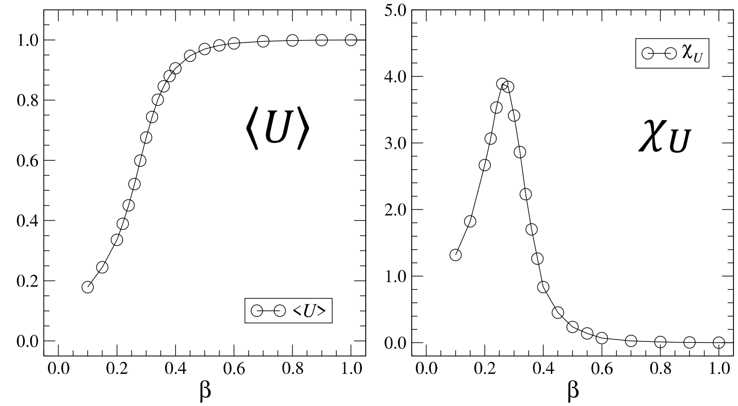
  Magnifico-Felser-Silvi-Montangero, Nat.Commun.12(2021)3600

Z_3 gauge-Higgs model w/ TRG

- Results by the MC and TRG at vanishing chemical potential are consistent

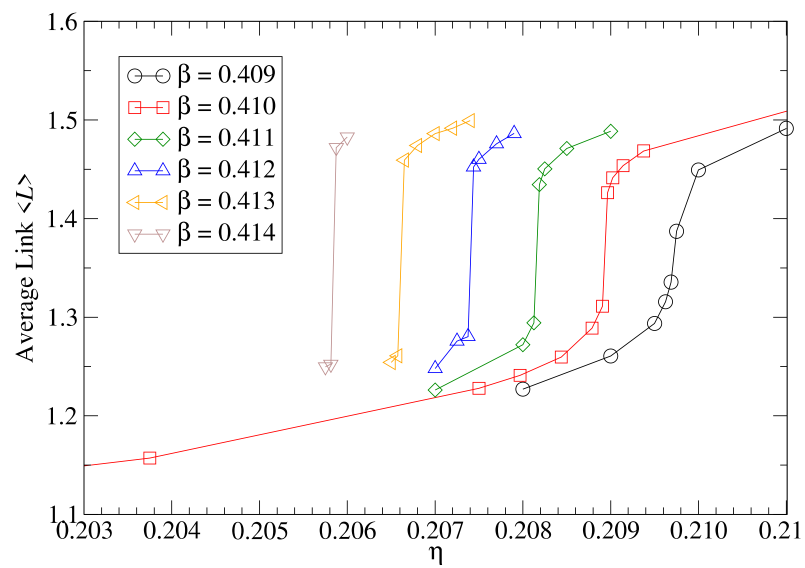


Gattringer-Schmidt, PRD86(2012)094506



SA+, JHEP10(2023)077

- First-order confinement-Higgs transition line and its endpoint are determined
  - Critical exponents are comparable to the mean-field ones also at finite density

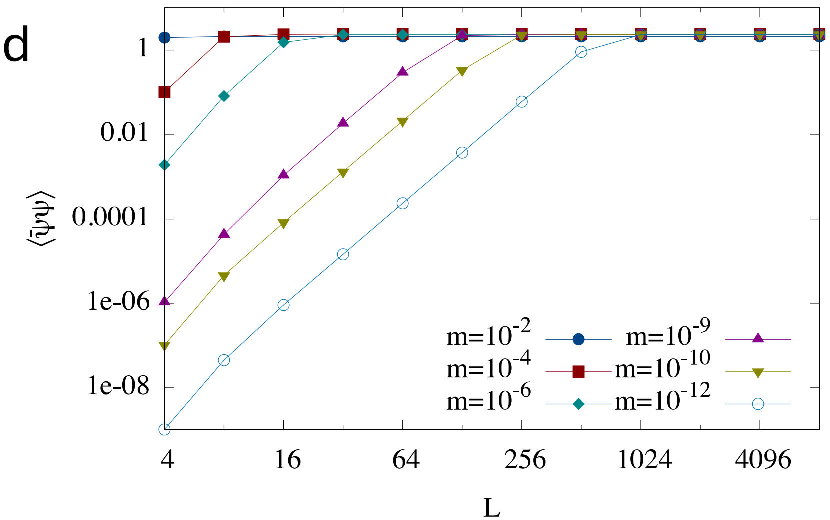


SA+, JHEP10(2023)077

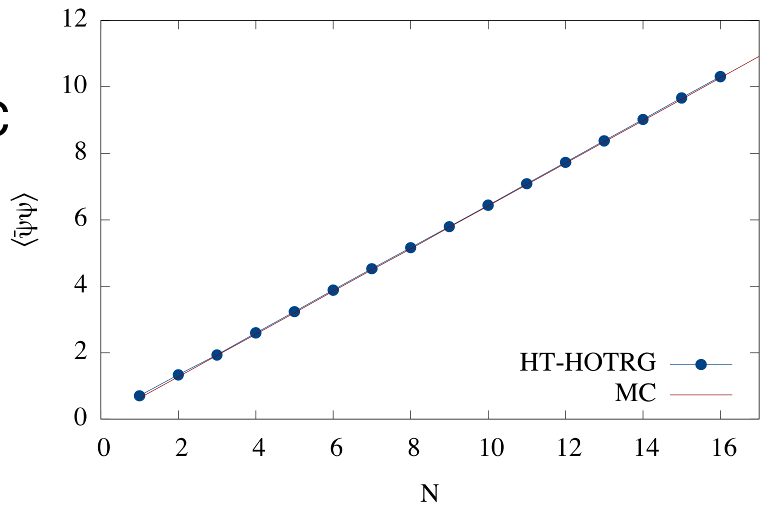
Infinite-coupling U(N) model w/ TRG

Milde+, Lattice2021

- Investigation of chiral symmetry breaking
  - Staggered fermion theory is reformulated as a dimer-monomer system
  - Infinite-volume limit and chiral limit
  - Chiral symmetry seems to be spontaneously broken in these limits
- Comparison with the MC simulation
  - Chiral condensate in the large-N limit
  - Difference is less than 1% btw TRG and MC



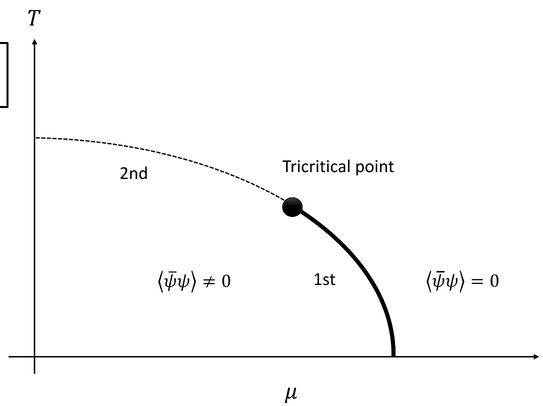
	$\lim_{N\to\infty} \frac{\langle\bar{\psi}\psi\rangle}{N}$
HT-HOTRG (D=20)	0.637(2)
Metropolis	0.64279(1)



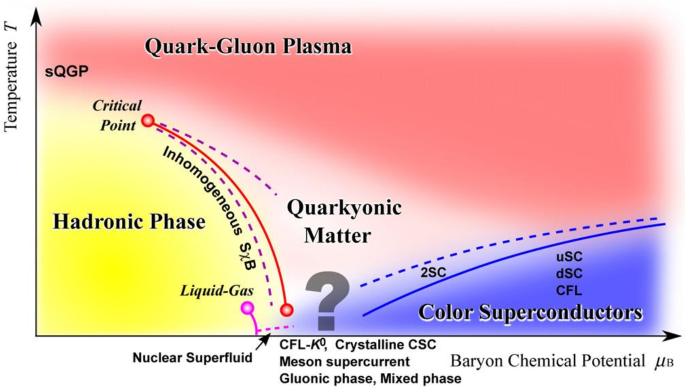
Nambu-Jona–Lasinio model w/ TRG

- NJL model at finite density is an effective theory of the QCD at finite density

NJL

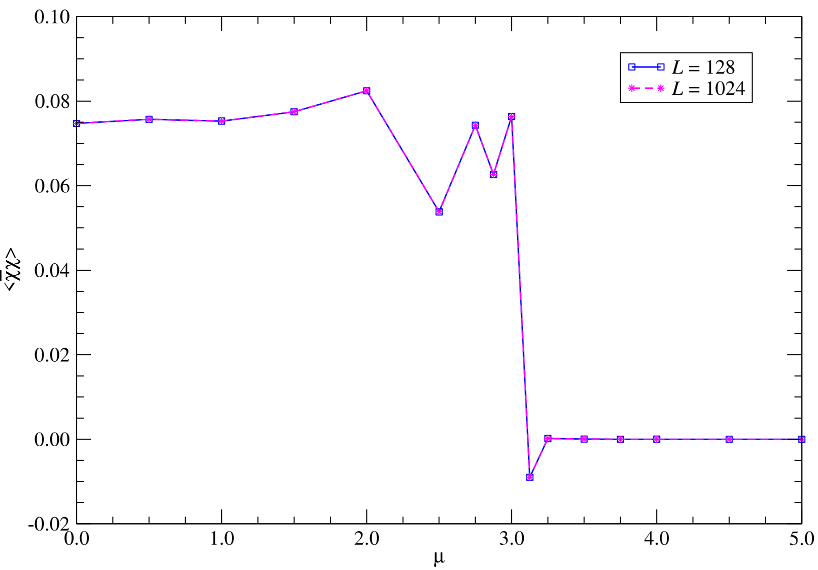


QCD



Fukushima-Hatsuda, Rept. Prog. Phys. 74(2011)014001

- **TRG has confirmed that the chiral symmetry is restored in the cold dense regime, where the MC is hindered by the severe sign problem**
  - Pressure and number density have also been obtained

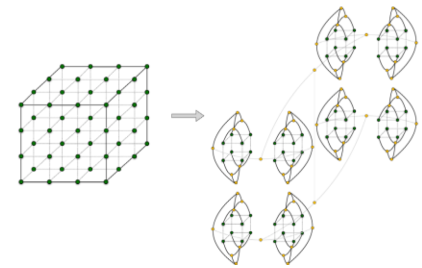


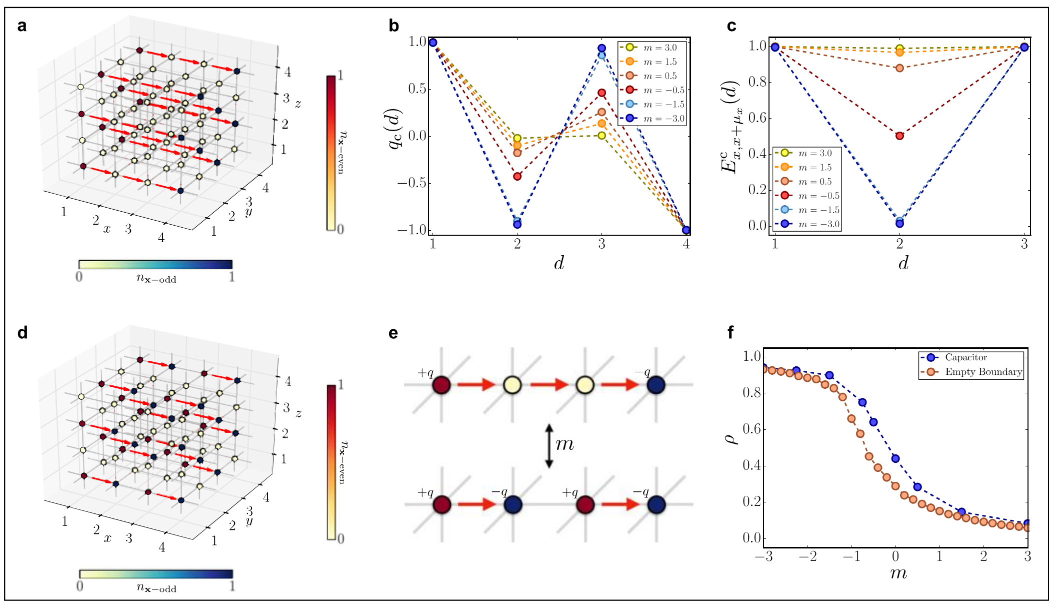
SA+, JHEP01(2021)121

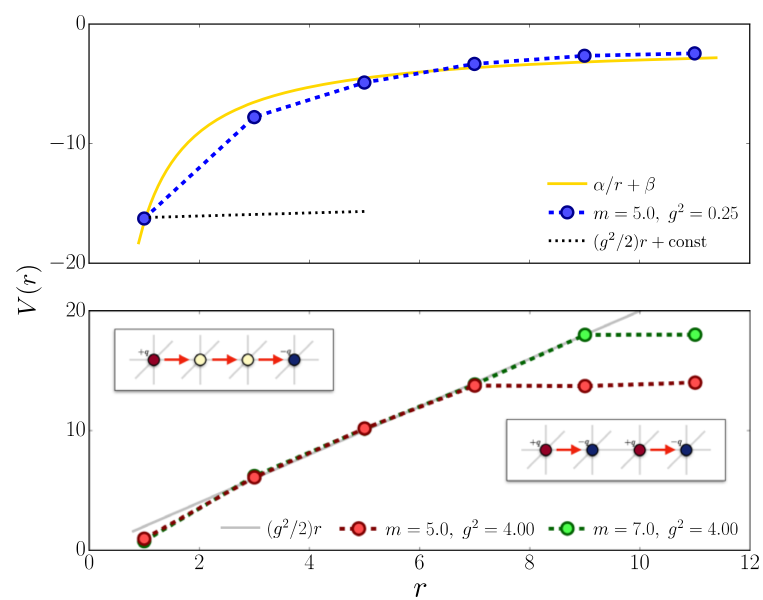
QED w/ TTN

Magnifico+, Nat.Commun.12(2021)3600

- The first TTN simulation of the ground state of the (3+1)D compact QED
  - Confinement properties are investigated
    - **Coulomb potential** in the weak coupling regime
    - **Linear potential** in the strong coupling regime







Status of TN in the higher dimensions

- TRG, TNS, and GGPEPS have been applied to the 3D, (2+1)D systems
  - Starting from the simplest spin models and proceedings to applications to non-Abelian gauge theories
  - Relatively large bond dimensions are available in TRG and TTN
- TRG algorithms are most employed in numerical computations of 4D, (3+1)D systems
  - Since the tensor contraction is the bottleneck in several TRG algorithms, parallelization can reduce the execution time
- Parallel computing methods specialized for individual algorithms are being developed

SA-Kuramashi-Yamashita-Yoshimura, Lattice2019
Sakurai-Yamashita, CPC278(2022)108423
Sun-Shirakawa-Yunoki, PRB110(2024)08514
Magnifico-Cataldi-Rigobello-Majcen-Jaschke-Silvi-Montangero, arXiv:2407.03058

Future perspectives

- The Hamiltonian- and Lagrangian-based TN algorithms share the problem of how to regularize or deal with the **non-Abelian DoFs**
  - Not only in classical TN computations but also in quantum computations. There is potential for mutual development
  - GGPEPS is providing a new approach toward this issue
- Need to develop the schemes to analyze numerical results that include **finite bond dimension effect**
  - Lessons form conformal field theories

Tagliacozzo-Oliveira-Iblisdir-Latorre, PRB78(2008)024410
Pollmann-Mukerjee-Turner-Moore, PRL102(2009)255701
Ueda-Oshikawa, PRB108(2023)024413
Huang-Chan-Kao-Chen, PRB107(2023)205123

- New TN algorithms by combining other methods
  - TN + Machine Learning

Liao-Liu-Wang-Xiang, PRX9(2019)031041
Chen-Gao-Guo-Liu-Zhao-Liao-Wang-Xiang-Li-Xie, PRB101(2020)220409
Jha-Samlodia, CPC294(2024)108941

  - TN + Monte Carlo
    - From the systematic error to statistical one

Ferris, arXiv:1507.00767
Huggins-Freeman-Stoudenmire-Tubman-Whaley, arXiv:1710.03757
Zohar-Cirac, PRD97(2018)034510
Arai-Ohki-Takeda-Tomii, PRD107(2023)114515
Todo, 18th Extreme Universe Colloquium (2023)
Chen-Guo-Zhang-Zhang-Deng, arXiv:2409.06538