# Incompressible supersolid in a dipolar condensate of interlayer excitons

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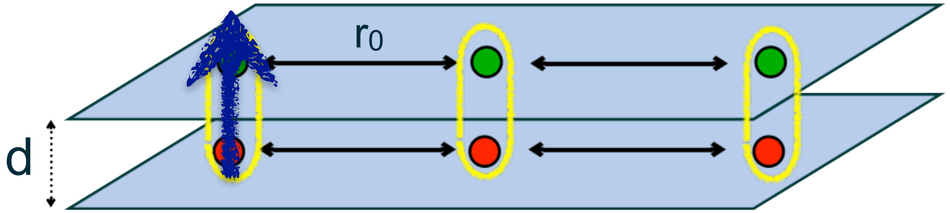
Research Foundation Flanders
Opening new horizons

Universiteit Antwerpen

# ELECTRON-HOLE PHASE DIAGRAM (T = 0)

$$\langle V_{ee} \rangle = \langle V_{hh} \rangle \propto \frac{1}{r_0}$$

$$\langle V_{eh} \rangle \propto -\frac{1}{d}$$




$a_B$: effective Bohr radius

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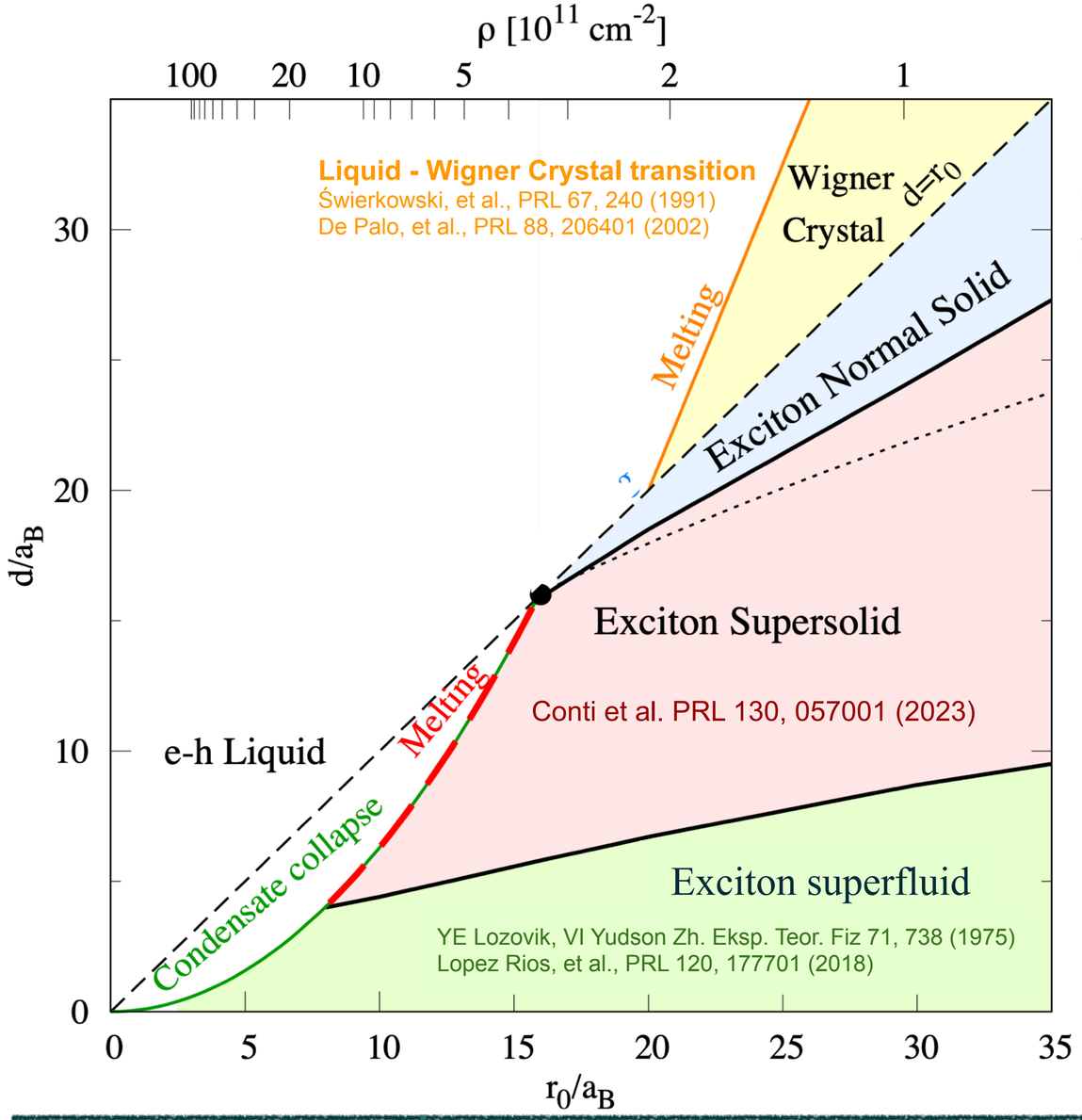


**Exciton liquid-solid transition**
Astrakharchik et al. PRL 98, 060405 (2007);
Böning, et al., PRB 84, 075130 (2011)

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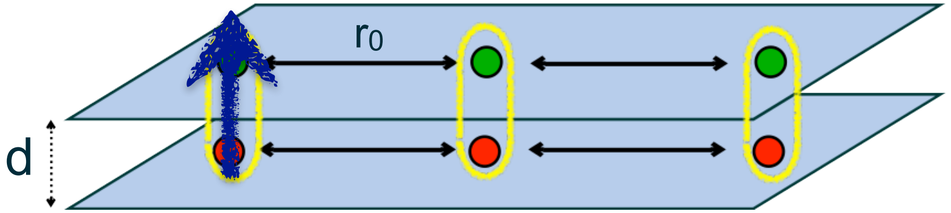


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# THEORY FOR EXCITON SUPERSOLID

The Hamiltonian is,

$$H = \int d^2\mathbf{r}\, \Psi^\dagger(\mathbf{r}) \left(-\frac{\hbar^2\nabla^2}{2M_{\mathrm{X}}}\right) \Psi(\mathbf{r}) + \frac{1}{2}\iint d^2\mathbf{r} d^2\mathbf{r}'\, \Psi^\dagger(\mathbf{r})\Psi^\dagger(\mathbf{r}')V_{\mathrm{XX}}(\mathbf{r}'-\mathbf{r})\Psi(\mathbf{r})\Psi(\mathbf{r}')$$

$$V_{XX}(r) = \frac{2\, e^2}{4\pi\epsilon}\left(\frac{1}{r} - \frac{1}{\sqrt{r^2+d^2}}\right)$$

Exciton-Exciton interaction (e-e, h-h, e-h Coulomb interactions)

- It is **purely REPULSIVE**!
- $d \ll r_0 \rightarrow V_{XX} \sim \frac{d^2}{r^3}$
- $d \gg r_0 \rightarrow V_{XX} = V_{ee} \sim \frac{1}{r}$

# GROSS PITAEVSKII EQUATION

We define the time-independent Gross-Pitaevskii equation (GPE) for $\Psi(\mathbf{r}, t) = \Psi(\mathbf{r}) e^{-i\mu t/\hbar}$

- For an exciton condensate with **non-local exciton-exciton interaction**.

$$-\frac{\hbar^2 \nabla^2}{2M_X}\Psi(\mathbf{r}) + \left( \int d^2\mathbf{r}' \, V_{XX}(\mathbf{r}-\mathbf{r}') \, |\Psi(\mathbf{r}')|^2 \right) \Psi(\mathbf{r}) = \mu \, \Psi(\mathbf{r}) \, .$$

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This is still an incomplete picture:

- it does not include **exciton-exciton correlations**
- it incorrectly includes interaction of an exciton on a site with itself (**Self interaction**)

# (i) TWO BODY CORRELATIONS

We work at very low exciton density: the short-range two-body correlations are known to be strong.

The exciton pair correlation function $g(\mathbf{r})$ vanishes across a region of small $r$.

[G. E. Astrakharchik, J. Boronat, I. L. Kurbakov, and Yu. E. Lozovik, Phys. Rev. Lett. 98, 060405 (2007)]

$V_{XX}^{eff}(\mathbf{r}) = g(\mathbf{r})V_{XX}(\mathbf{r})$ with $g(\mathbf{r})=0$ when $r < R_{HardCore}$.

We include the effect with an effective Hamiltonian.

$$\mathscr{H}^{\text{eff}} = \int d^2\mathbf{r}\, \Psi^\dagger(\mathbf{r})\left(-\frac{\hbar^2\nabla^2}{2M_X}\right)\Psi(\mathbf{r}) + \frac{1}{2}\iint d^2\mathbf{r}d^2\mathbf{r}'\, \Psi^\dagger(\mathbf{r})\Psi^\dagger(\mathbf{r}')t_{XX}(\mathbf{r}'-\mathbf{r})\Psi(\mathbf{r})\Psi(\mathbf{r}')$$

The local t-matrix $t_{XX}(\mathbf{r})$ (multiple short-range two-body scattering) vanishes for $r < R_{HardCore}$.

# (ii) THE TRUE BOSE SOLID

For a complete basis with fixed lattice sites *i* hosting a mutually orthogonal localized orbital $\phi_i(r)$

The effective Hamiltonian can be written as

$$\mathscr{H}^{\text{eff}} = \sum_{i,j} c_i^\dagger T_{ij}\, c_j + \frac{1}{2} \sum_{i,i',j,j'} c_i^\dagger c_{i'}^\dagger\, U^{XX}_{i,i',j,j'}\, c_j c_{j'}$$

$$T_{ij} = \int d^2\mathbf{r}\, \phi_i(\mathbf{r}) \left( -\frac{\hbar^2 \nabla^2}{2M_X} \right) \phi_j(\mathbf{r})$$

$$U^{XX}_{i,i',j,j'} = \iint d^2\mathbf{r}\, d^2\mathbf{r}'\, \phi_i(\mathbf{r})\, \phi_{i'}(\mathbf{r}')\, t_{XX}(\mathbf{r}'-\mathbf{r})\, \phi_j(\mathbf{r})\, \phi_{j'}(\mathbf{r}')$$

with

$$c_i^\dagger = \int d^2r\, \phi_i(\mathbf{r}) \Psi^\dagger(\mathbf{r}); \qquad c_i = \int d^2r\, \phi_i(\mathbf{r}) \Psi(\mathbf{r})$$

The task is to self-consistently construct a “true Bose solid” ground state of the form,

$$|\Psi_0\rangle = \left( \prod_{i=1}^{N} c_i^\dagger \right) |\Psi_{\text{vac}}\rangle$$

[P. W. Anderson, Basic notions of condensed matter physics (Addison-Wesley, Reading, Mass., 1997)]

# (ii) EQUATIONS OF MOTION

The equations of motion for the particle $c_i^\dagger \,|\, \Psi_0\rangle$ and anti-particle $c_i \,|\, \Psi_0\rangle$ excitations:

$$\frac{dc_m^\dagger}{dt} = i[\mathscr{H}, c_m^\dagger] = -\, i \sum_i c_i^\dagger \, T_{im} - i\frac{1}{2}\sum_{ii'j'} c_i^\dagger c_{i'}^\dagger \left( U^{XX}_{i,i',m,j'} + U^{XX}_{i,i',j',m} \right) c_{j'}$$

$$\frac{dc_m}{dt} = i[\mathscr{H}, c_m] = i \sum_j T_{mj} \, c_j + i\frac{1}{2}\sum_{ij\neq j'} c_i^\dagger \left( U^{XX}_{m,i,j,j'} + U^{XX}_{i,m,j,j'} \right) c_j c_{j'}$$

**The particle and anti-particle excitation operators obey different equations of motion in this system:**

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**The particle and anti-particle excitation operators obey different equations of motion in this system:**

- $c_j \,|\, \Psi_0\rangle = 0$ since, with only one particle on site j, the site cannot be emptied twice.
- $c_i^\dagger \,|\, \Psi_0\rangle \neq 0$ since for bosons two particles can be added to site j.

# (ii) PARTICLE EXCITATIONS

- We apply Hartree-Fock
- The particle eigenstate equation for the site orbital $\phi_j(\mathbf{r}_0)$

$$E_j^{(p)}\phi_j(\mathbf{r}_0) = -\,i\left(-\frac{\hbar^2\nabla^2}{2M_X}\right)\phi_j(\mathbf{r}_0) - i\;V_{eff}(\mathbf{r}_0)\,\phi_j(\mathbf{r}_0) - i\int d^2\mathbf{r}\;A(\mathbf{r}-\mathbf{r}_0)\,\phi_j(\mathbf{r})$$

$$V_{eff}(\mathbf{r}) = \int d^2\mathbf{r}'\,t_{XX}(\mathbf{r}-\mathbf{r}')\,\langle\Psi^\dagger(\mathbf{r}')\Psi(\mathbf{r}')\rangle \qquad A(\mathbf{r}-\mathbf{r}') = t_{XX}(\mathbf{r}-\mathbf{r}')\,\langle\Psi^\dagger(\mathbf{r})\Psi(\mathbf{r}')\rangle$$

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- Periodic effective potentials: solution for an added particle is identical for each j site and is not localized.
- **The eigenvalue $E_j^{(p)}$ does not depend on the site index**: solutions are the complete set of Bloch waves.
- $E_j^{(p)}$ bounded from below by zero, there can be no bound state solutions for particle excitations.

# (ii) ANTI-PARTICLE EXCITATIONS

- The anti-particle eigenstate equation for the site orbital $\phi_j(\mathbf{r}_0)$

$$E_j^{(a)}\phi_j(\mathbf{r}_0) = \left(-\frac{\hbar^2\nabla^2}{2M_X}\right)\phi_j(\mathbf{r}_0) + i\left[V_{eff}(\mathbf{r}_0) - \boxed{\overline{\Delta V_j}(\mathbf{r}_0)}\right]\phi_j(\mathbf{r}_0) + i\int d^2\mathbf{r}'\,\left[A(\mathbf{r}'-\mathbf{r}_0) - \boxed{\overline{\Delta A_j}(\mathbf{r}'-\mathbf{r}_0)}\right]\phi_j(\mathbf{r}')$$

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$$\overline{\Delta V_j}(\mathbf{r}) = -\int_{\mathbf{r}'\in j} d^2\mathbf{r}'\, t_{XX}(\mathbf{r}'-\mathbf{r})\,\langle\Psi^\dagger(\mathbf{r}')\Psi(\mathbf{r}')\rangle \quad \text{if } \mathbf{r}\in j \qquad \overline{\Delta A_j}(\mathbf{r}-\mathbf{r}') = -\,t_{XX}(\mathbf{r}'-\mathbf{r})\,\langle\Psi^\dagger(\mathbf{r})\Psi(\mathbf{r}')\rangle \quad \text{if } \mathbf{r}\in j$$

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- These **additional potentials** are localized on site j: behave like an attractive potential well on site j.
- **The eigenvalue $E_j^{(a)}$ depends on the site index.**

# (ii) INCOMPRESSIBLE SUPERSOLID

The differing equations of motion for anti-particles and particles lead to an **particle-antiparticle energy gap.**

If $V_{XX}$ is strong enough (large d), the potential well on each site can be sufficiently deep to form a bound states.

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If $V_{XX}$ is strong enough (large d), the potential well on each site can be sufficiently deep to form a bound states.

The **absence of self-interaction** for a particle on a site with itself leads to an **incompressible supersolid** (true Bose solid).

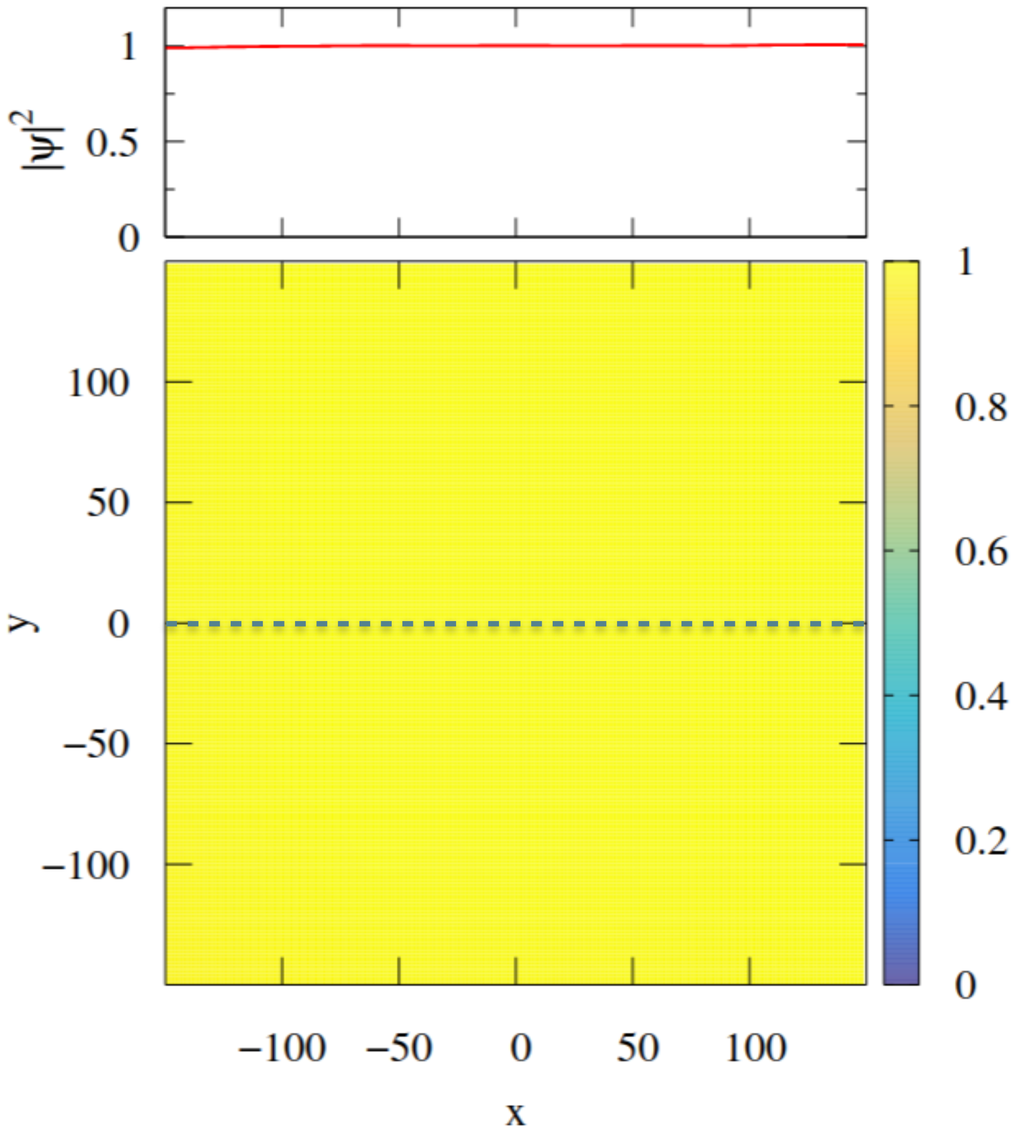
# EXTENDED GROSS PITAEVSKII EQUATION

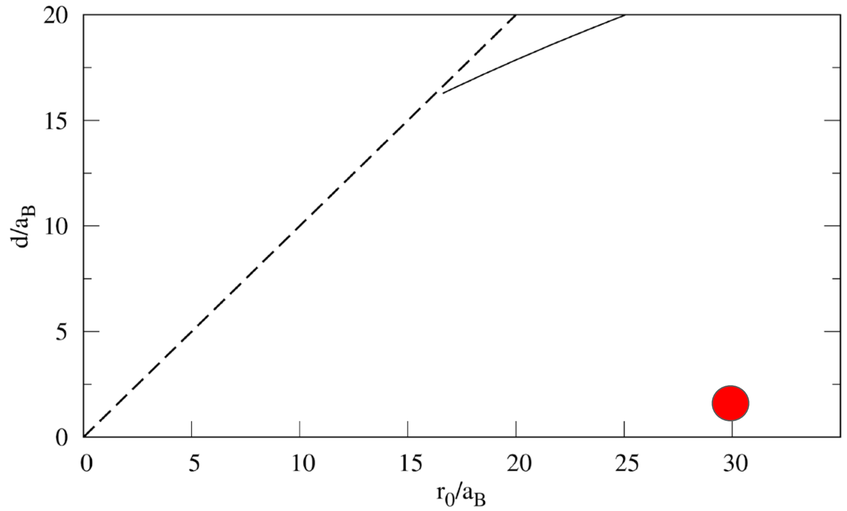
- For an exciton condensate with **non-local exciton-exciton interaction, correlations** and without **self-interaction**.

$$-\frac{\hbar^2 \nabla^2}{2M_X}\Psi(\mathbf{r}) + \left(\int d^2\mathbf{r}'\, t_{XX}(\mathbf{r}-\mathbf{r}')\,|\Psi(\mathbf{r}')|^2\right)\Psi(\mathbf{r}) - \boxed{\left(\int_{\mathbf{r}'\in j_{\mathbf{r}}} d^2\mathbf{r}'\, t_{XX}(\mathbf{r}-\mathbf{r}')\,|\Psi(\mathbf{r}')|^2\right)\Psi(\mathbf{r})} = \mu\,\Psi(\mathbf{r})$$

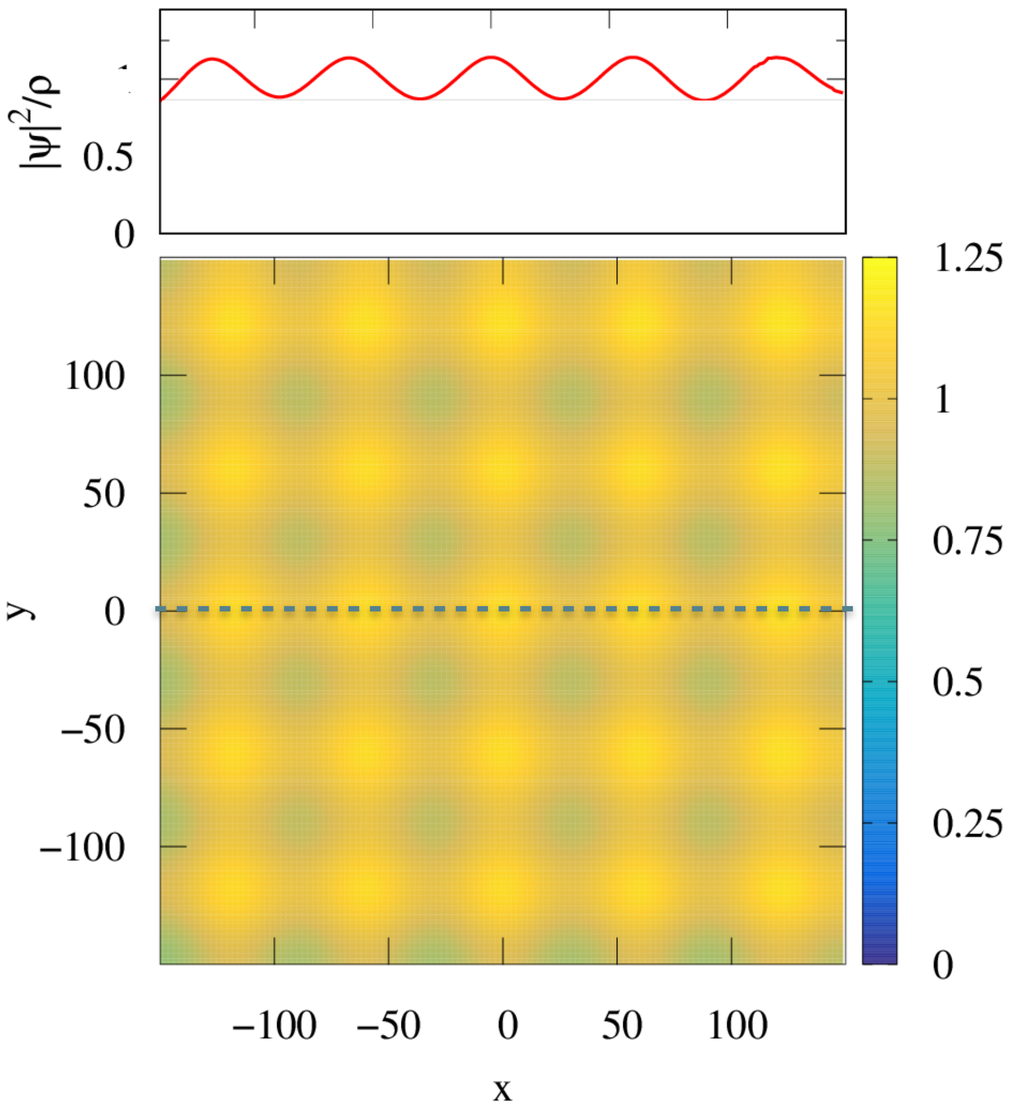
Self interaction at site $j_{\mathbf{r}}$

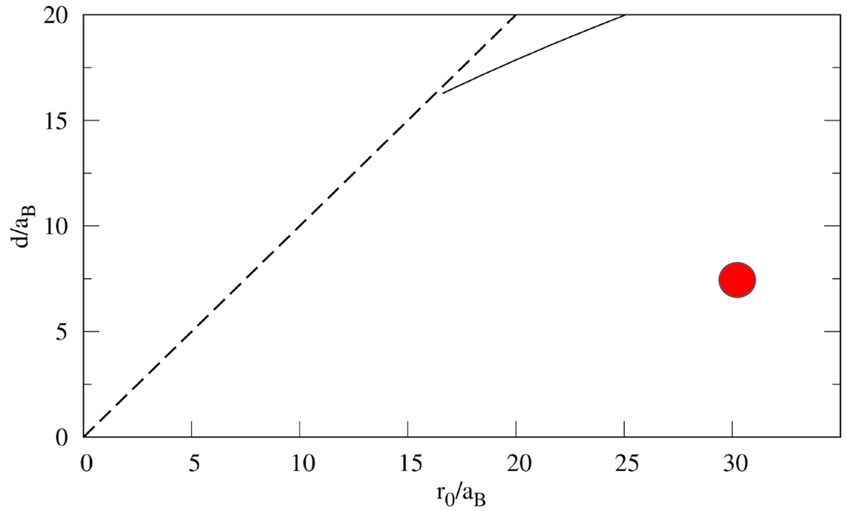
# GPE for d=1 Superfluid homogeneous solution



# GPE for d=8 Supersolid solution



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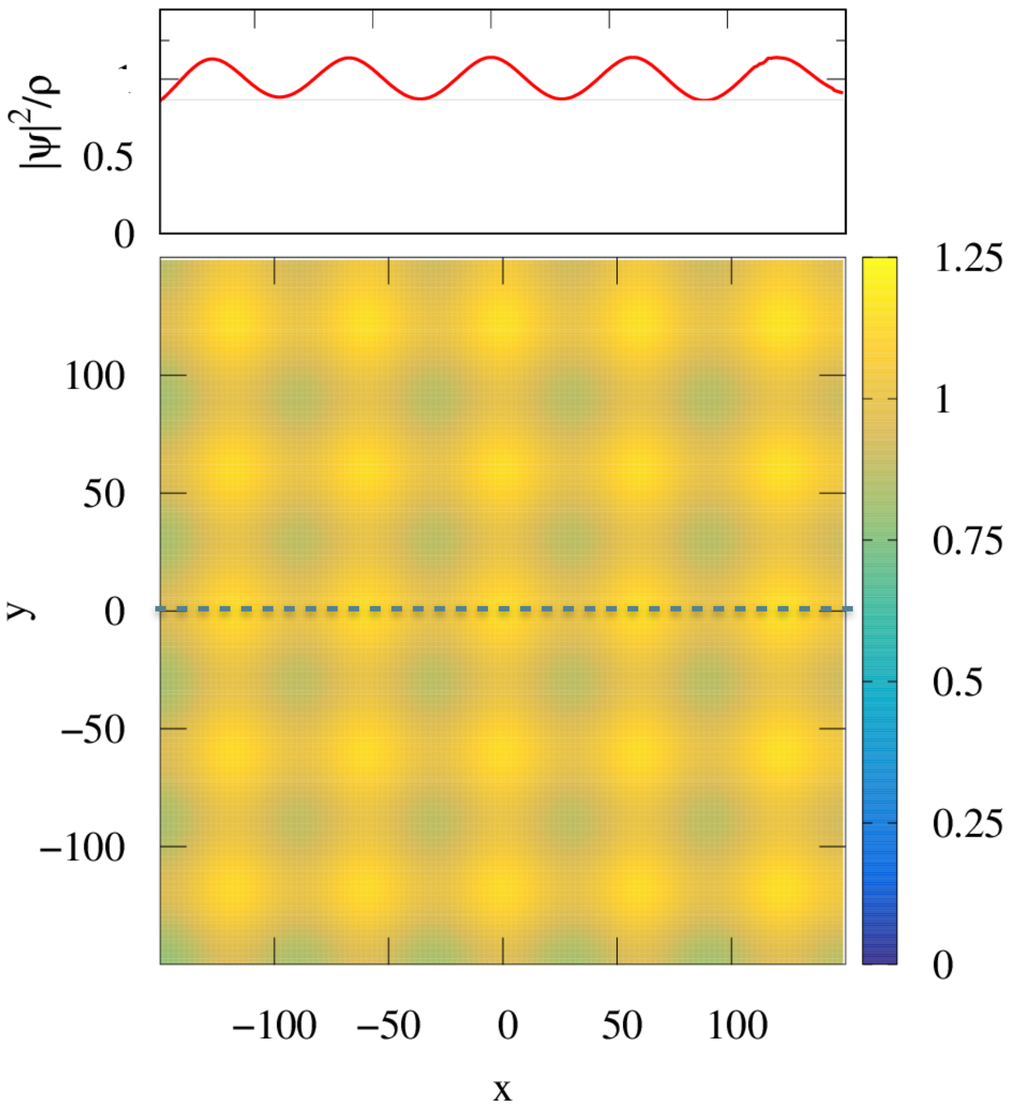


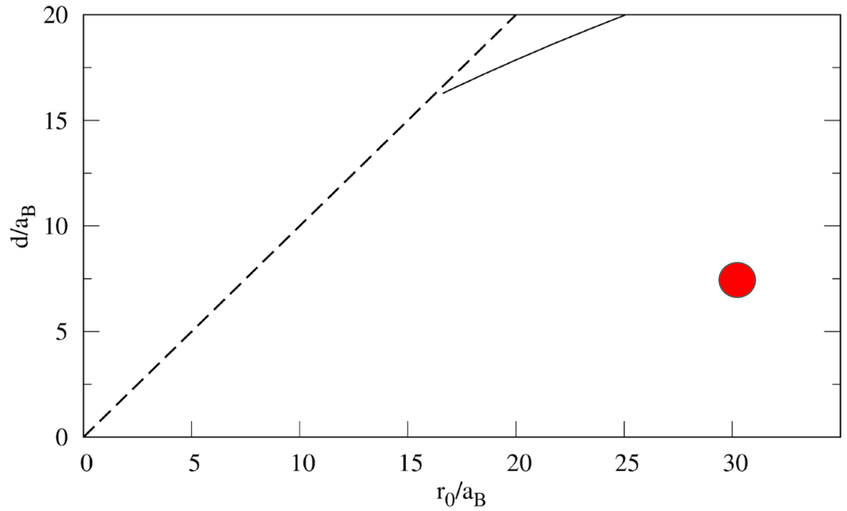


**Note this is not a density wave!**
**This is an incompressible supersolid**

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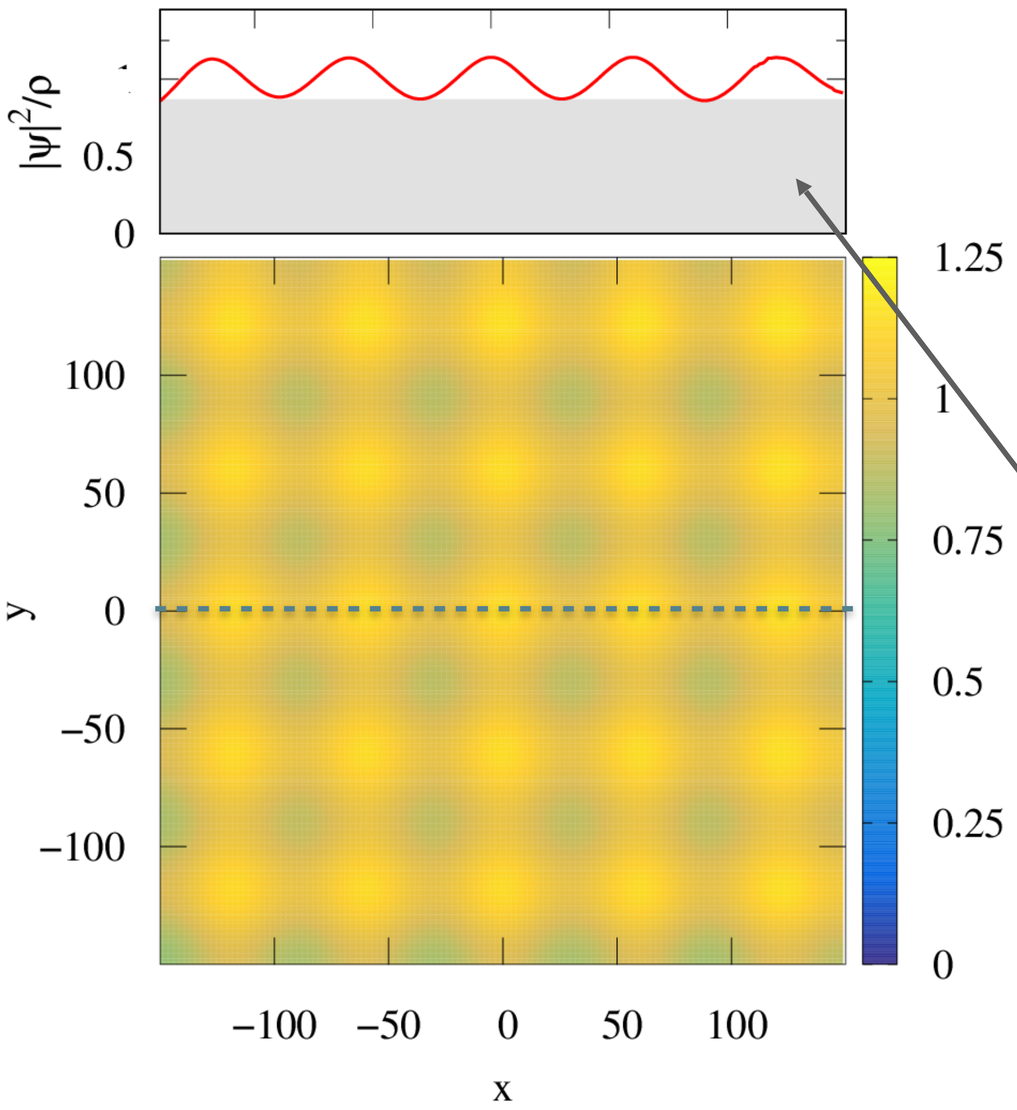
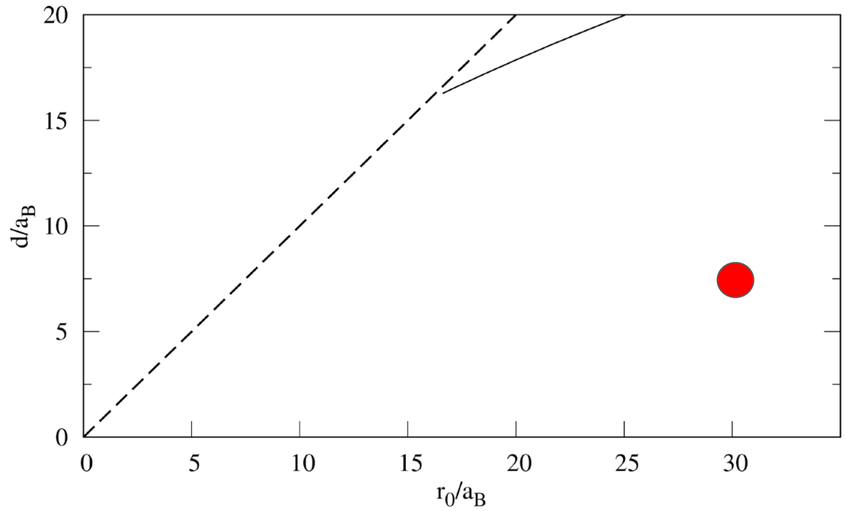
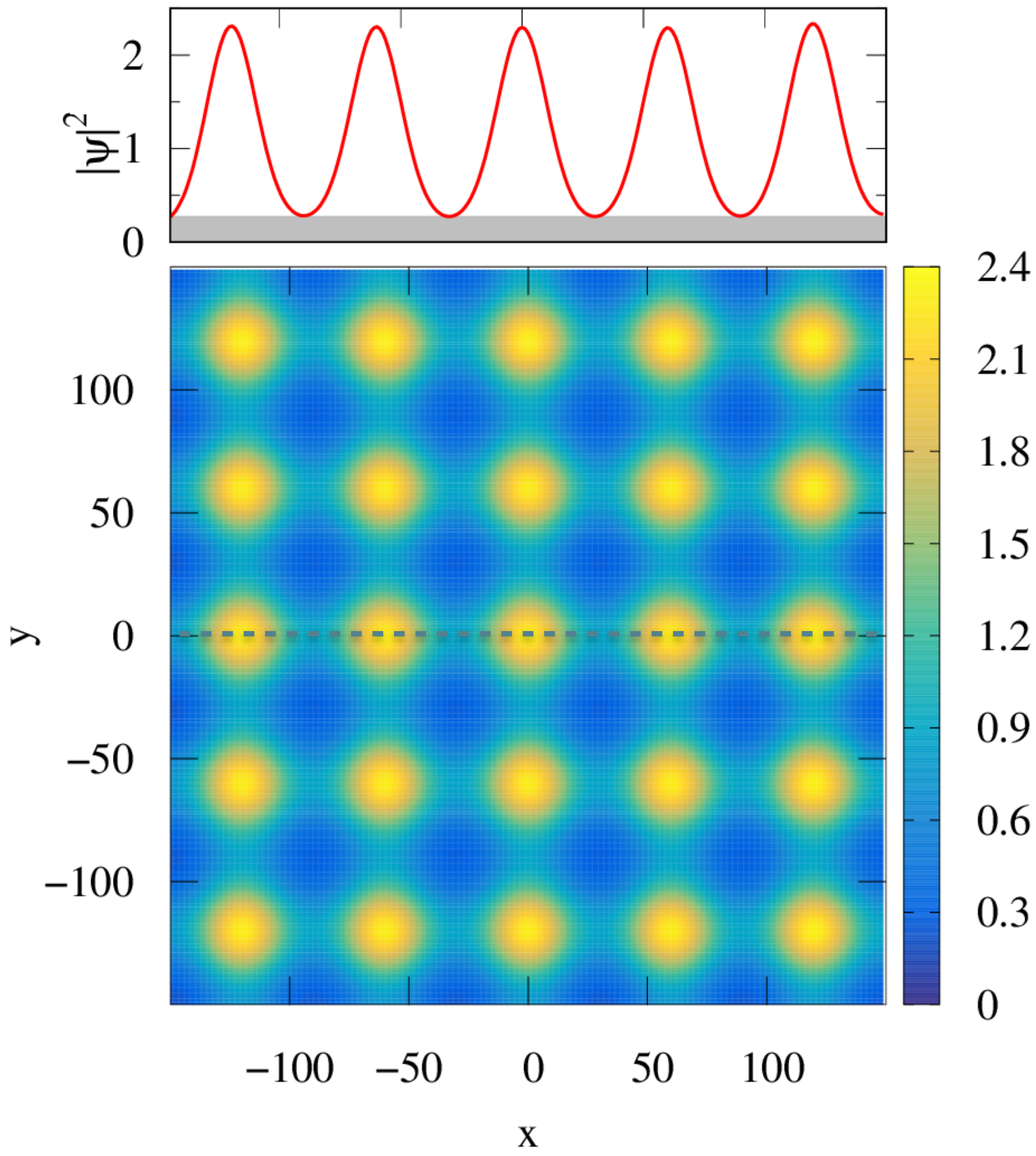
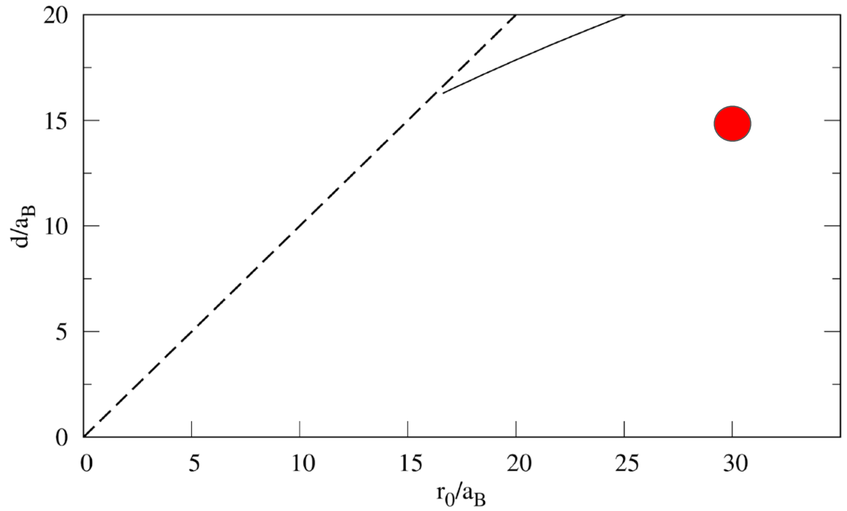



Grey area is a background contribution to the supersolid order parameter (strong Leggett exchange term).
[A. J. Leggett, Phys. Rev. Lett. 25, 1543(1970)]

# GPE for d=15 Supersolid solution



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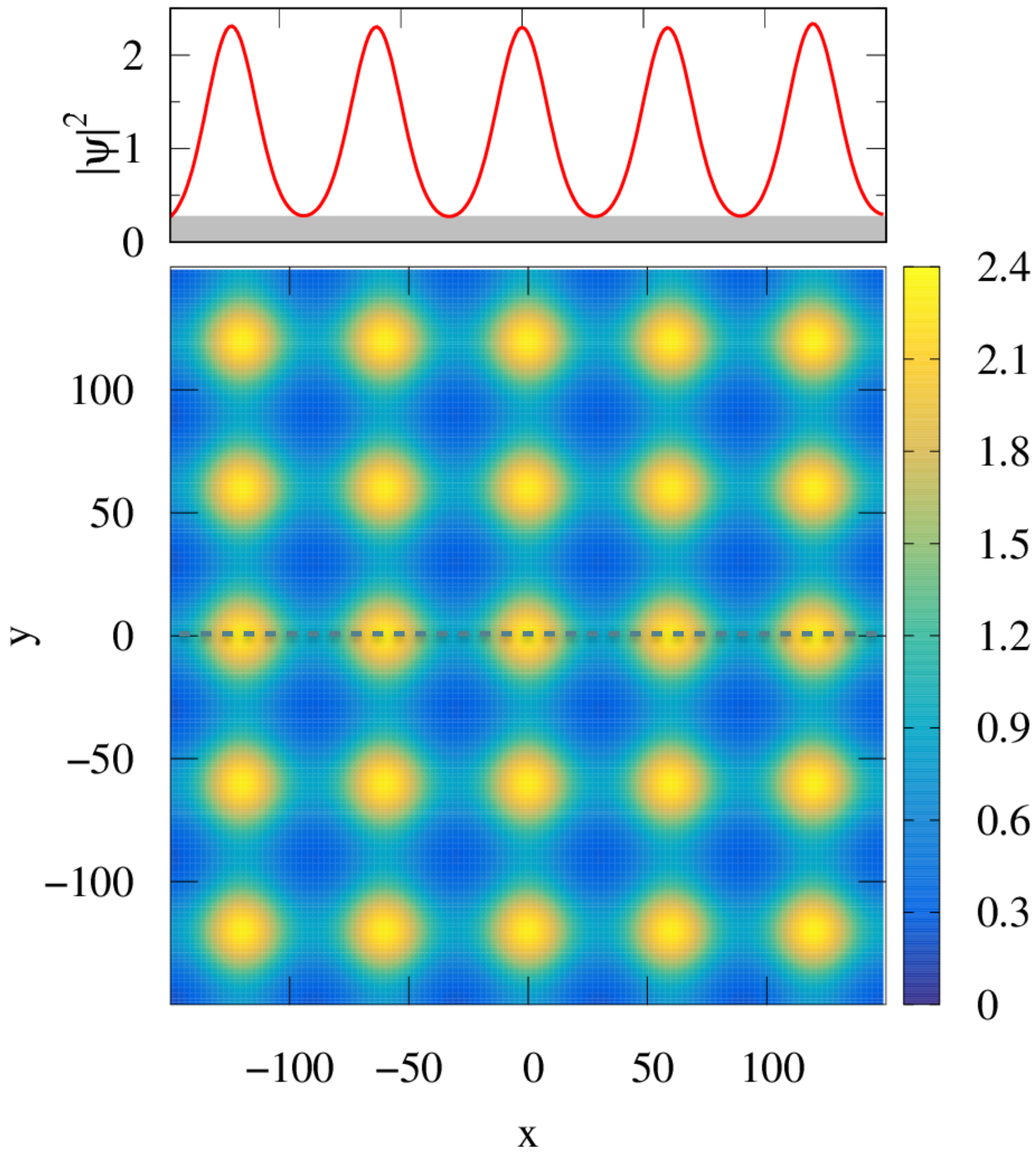
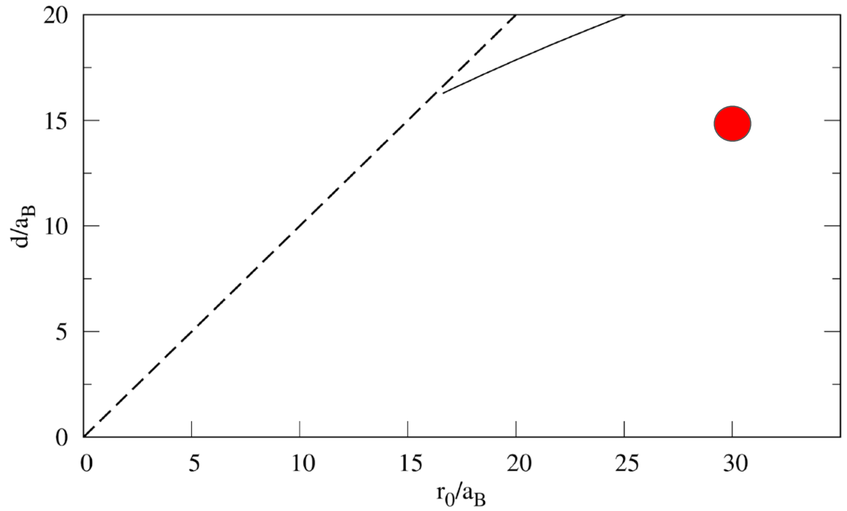
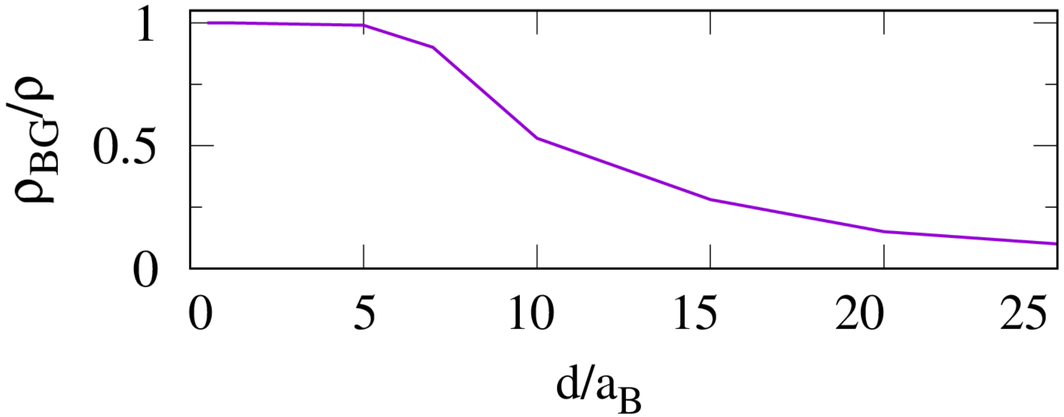
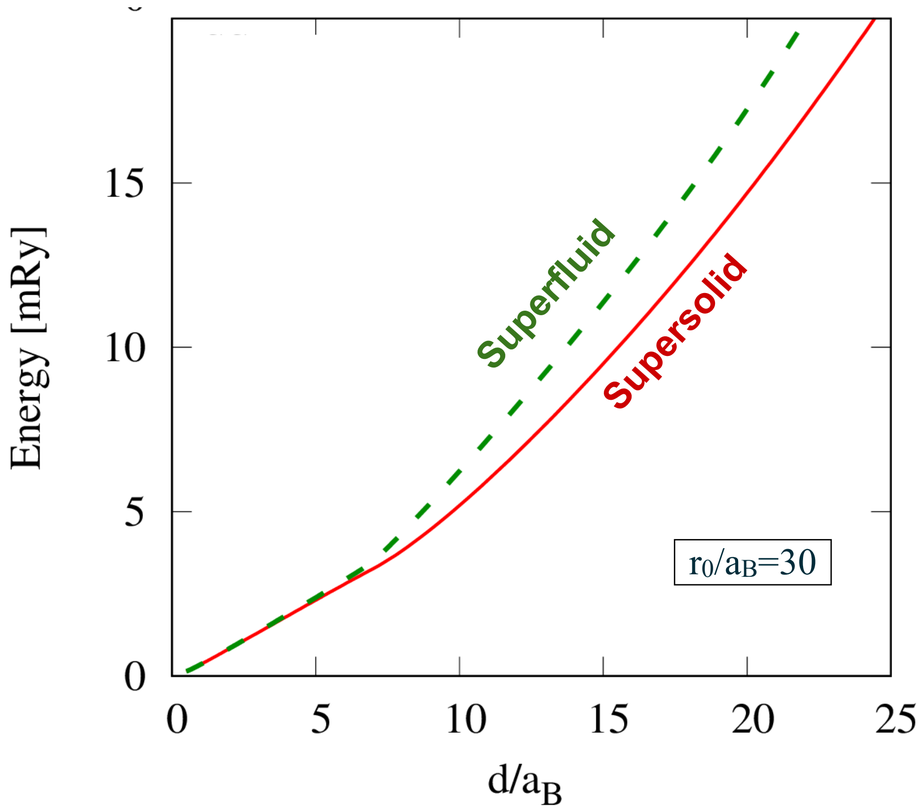


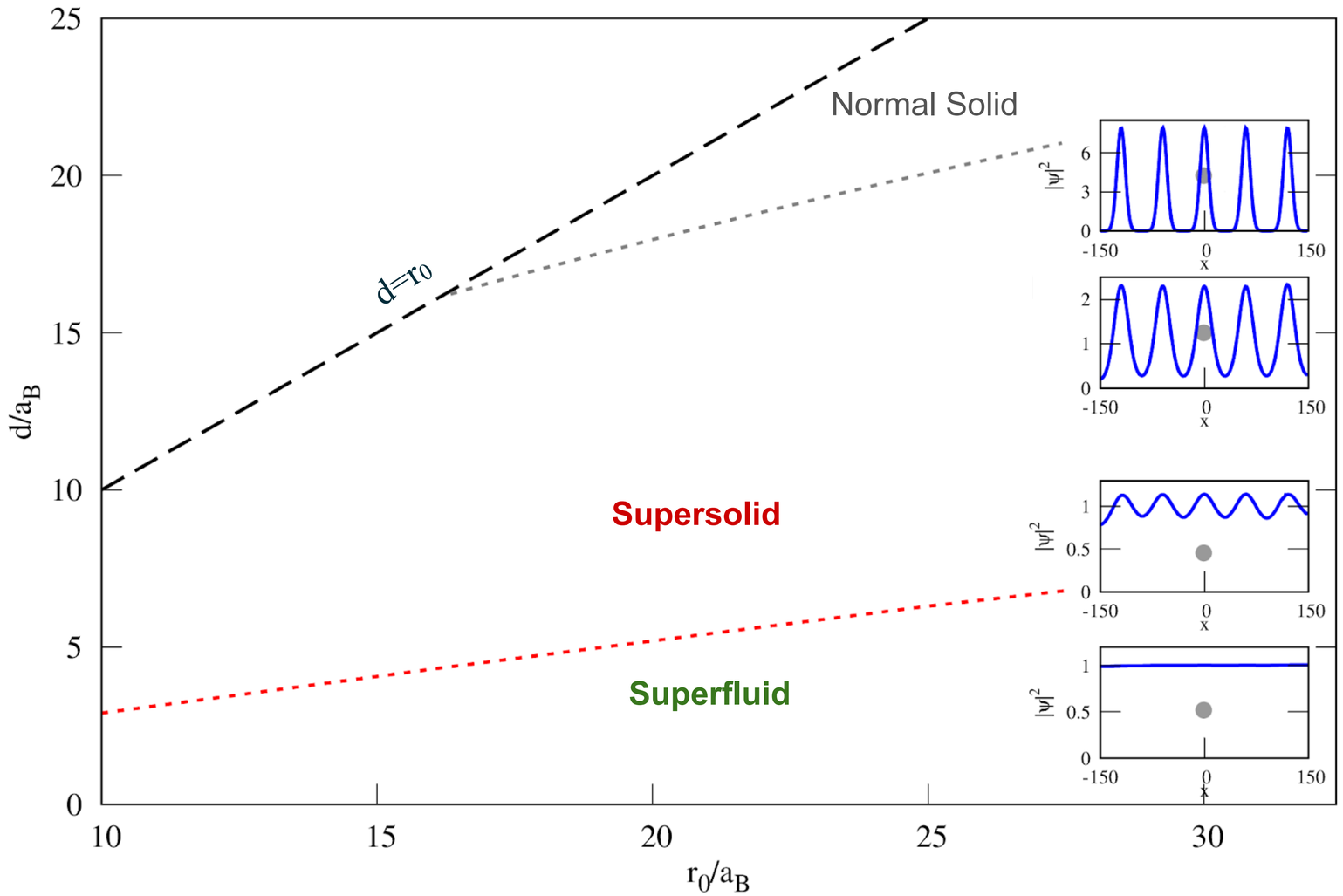




The background contribution (BG) in the Ψ(r) decreases by increasing d.



# SUPERFLUID TO SUPERSOLID TRANSITION




- For small d the GPE gives the superfluid solution.
- As d increases there is a **Superfluid to Supersolid transition**. [Conti et al. PRL 130, 057001 (2023)]

# MELTING OF THE SUPERSOLID

As $r_0$ decreases (density increases):

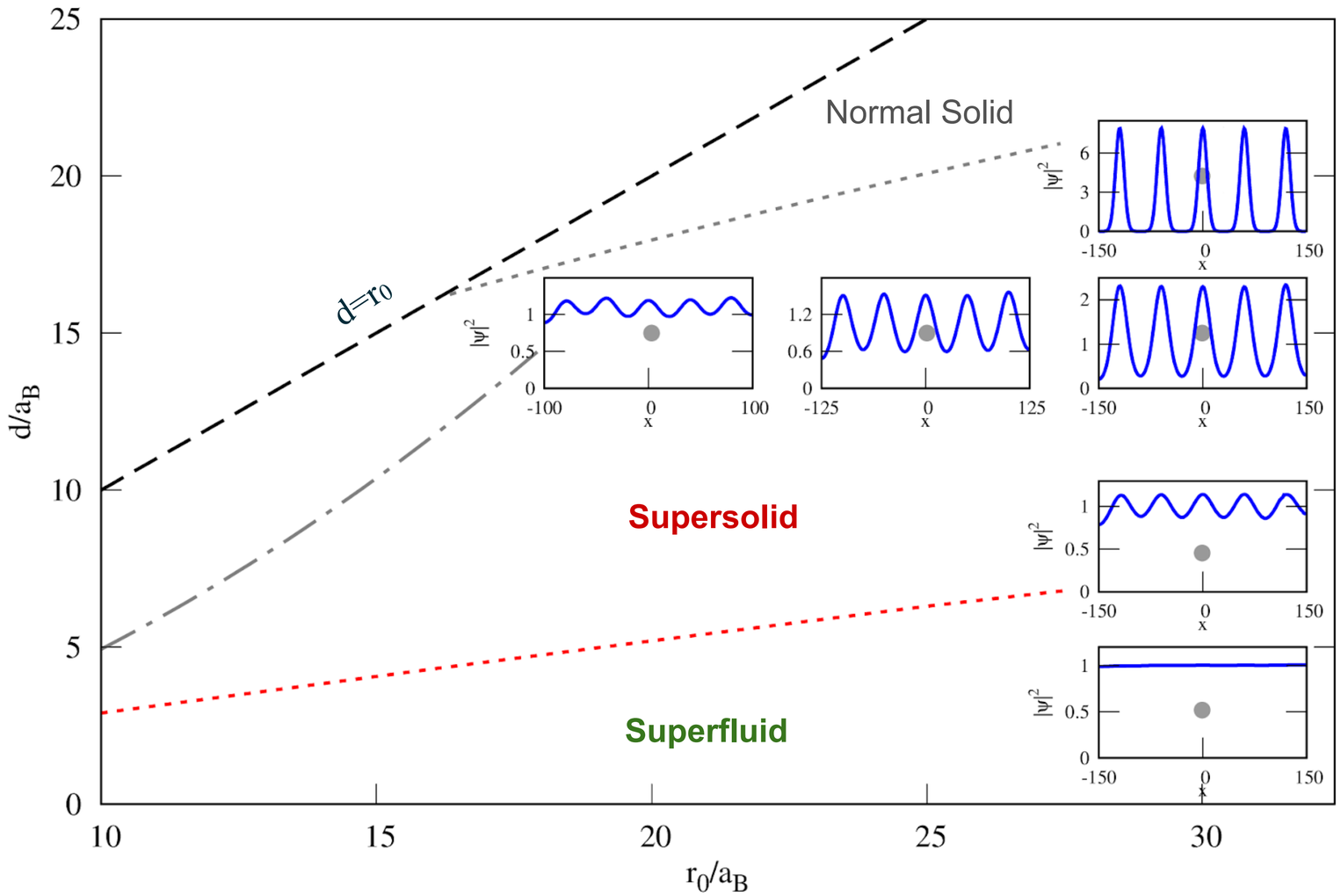
- $V_{xx}$ goes from dipolar-like to Coulomb-like interaction:
  ***$V_{xx}$* cannot support solidification.**

[J. Böning, et al. Phys. Rev. B 84, 075130 (2011)]

- Screening and intralayer correlations become stronger:
  **the quantum coherence collapses**.

[F. Pascucci et al. Phys. Rev. B **109**, 094512 (2024)]

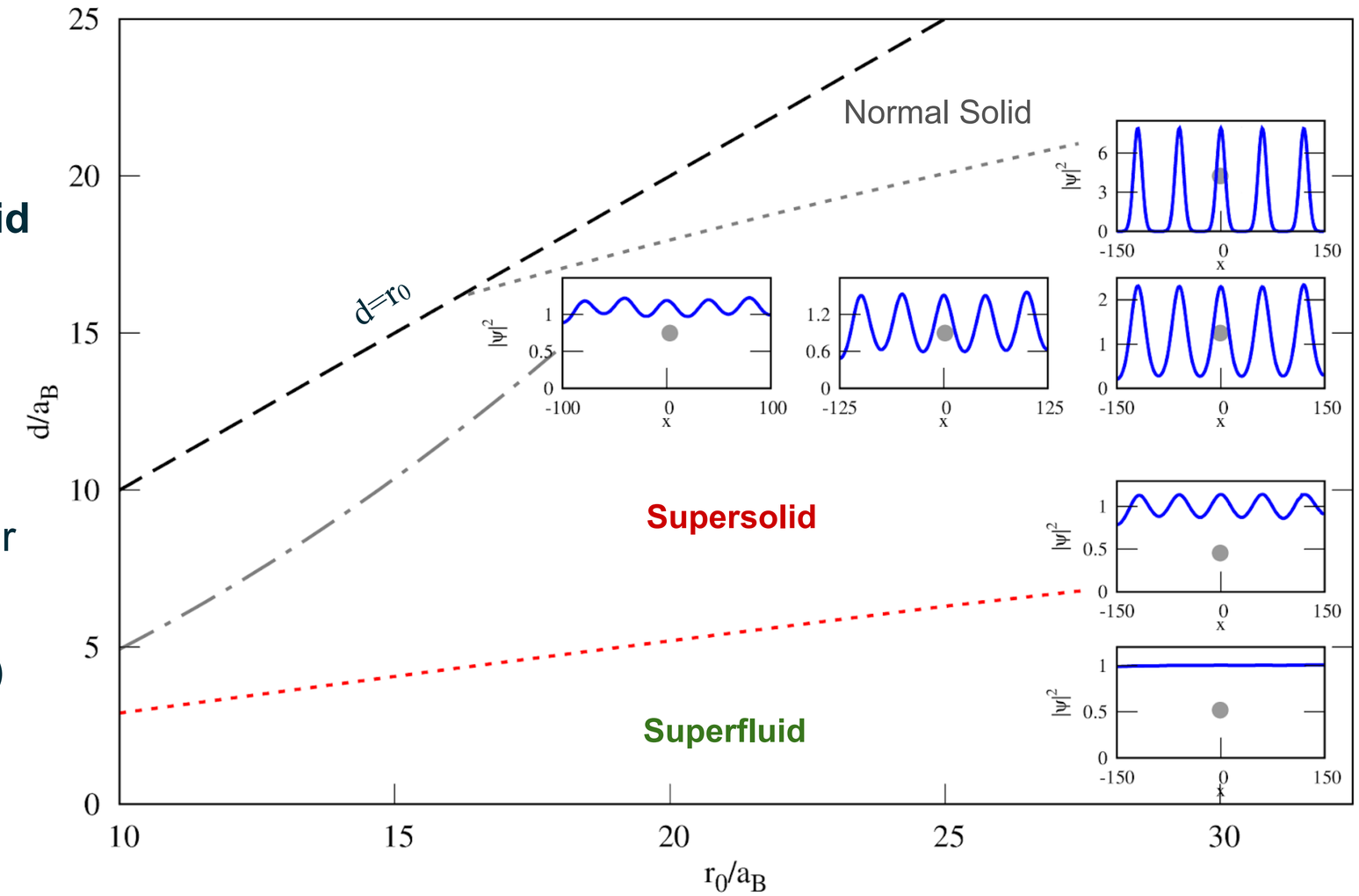
**Melting of the supersolid!**



# SUMMARY, but lots more to do

- **The absent self-interactions** for one particle on each site leads to an **incompressible supersolid**.
- GPE predicts a **superfluid to supersolid transition**.
- GPE well describes the evolution of the supersolid ground state throughout the phase diagram.
- We set the foundations to explore further supersolid properties, eg: vortices

(see David Neilson TALK Friday @09.30)



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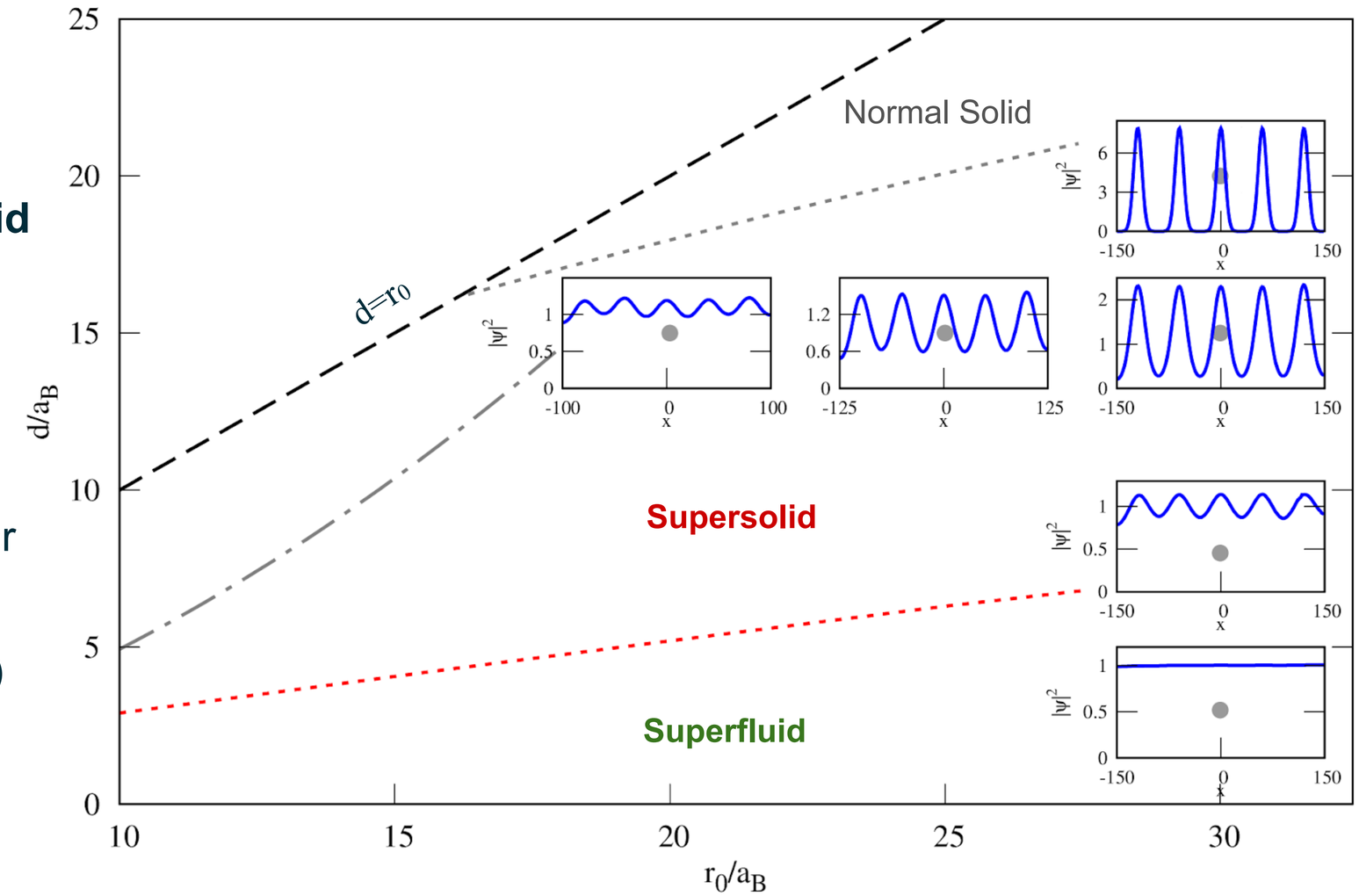


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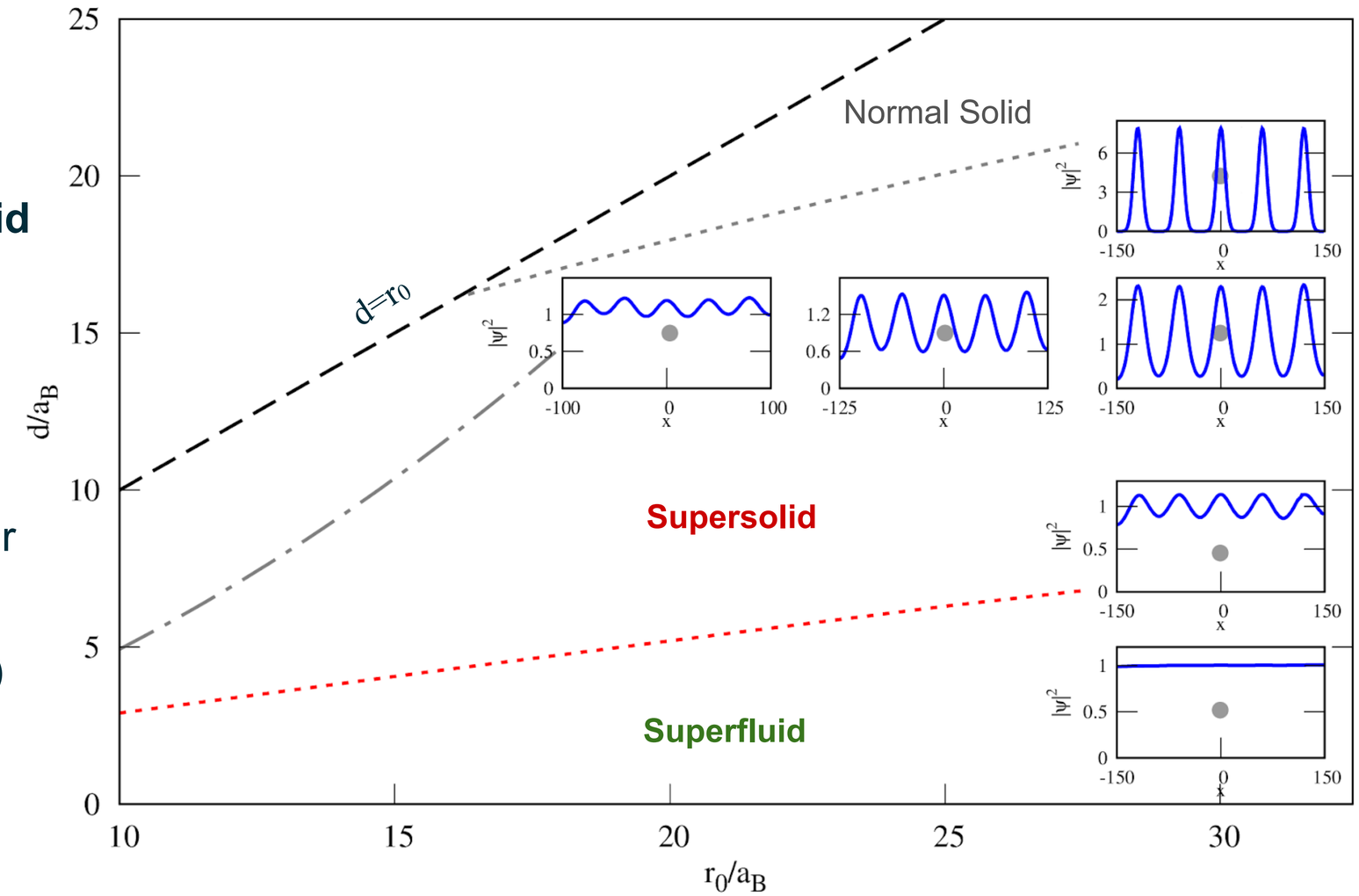

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