

# Quasi-steady state descriptions for photo-doped Mott insulators

Yuta Murakami (RIKEN CEMS)

Ref

YM, S. Takayoshi, T. Kaneko, Z. Sun, D. Golež, A. J. Millis and P. Werner, *Comm. Phys.* **5**, 23 (2022).

YM, S. Takayoshi, T. Kaneko, A. Läuchli and P. Werner, *Phys. Rev. Lett.* **130**, 106501 (2023).

**Review:** YM, D Golež, M Eckstein, P Werner, arXiv:2310.05201.



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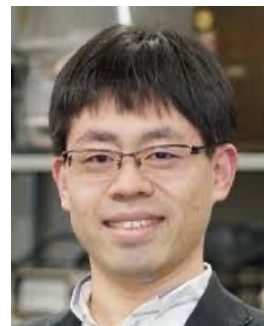
A. Läuchli

*Tsinghua Uni.*



Z. Suni

*Konan University*



S. Takayoshi

*Osaka University*



T. Kaneko



## Weak light excitation (Linear regime)



- ▷ Same frequencies of input and output
- ▷ Matter stays in equilibrium

## Strong light excitation (Nonlinear regime)



- ▷ Change in properties of output light  
ex) High-harmonic generation
- ▷ Change in properties of matter  
ex) Insulator-metal transition,  
light-induced superconductor

$$\text{Strong light} + \text{Matter} = \text{Control of properties of light and matter}$$

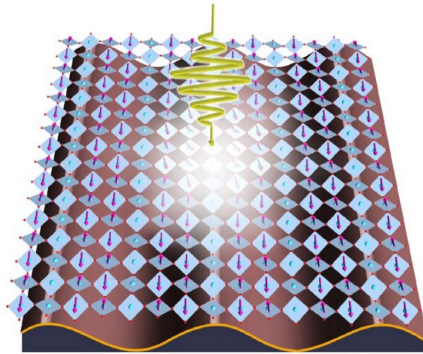
# Strong light-matter coupling and emergent phenomena

Rapid development on strong laser techniques in THz and mid-infrared regime

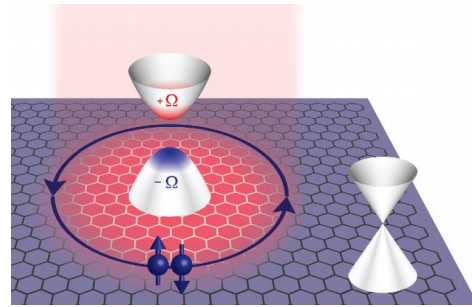


## Control of physical properties

Photo-induced phase transition, Photo-doping, Floquet engineering, nonlinear phononics, etc..



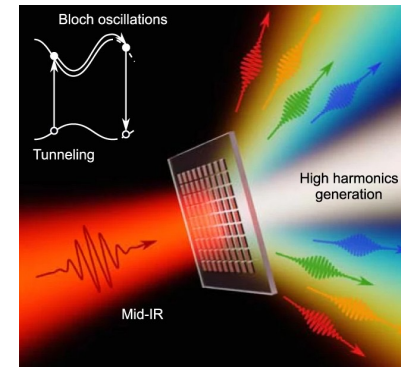
From S. Keiser



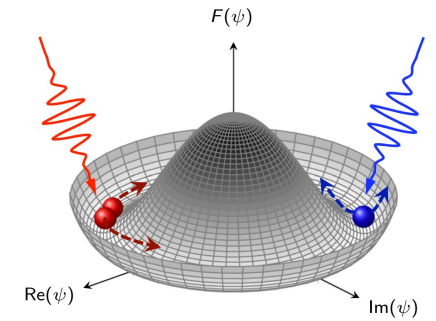
From J. McIver

## Intriguing optical responses

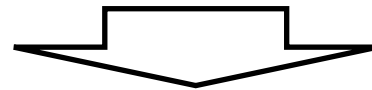
High-harmonic generation, Shift current, Higgs modes in superconductors, etc..



From D. Shilkin



From F. Gabriele

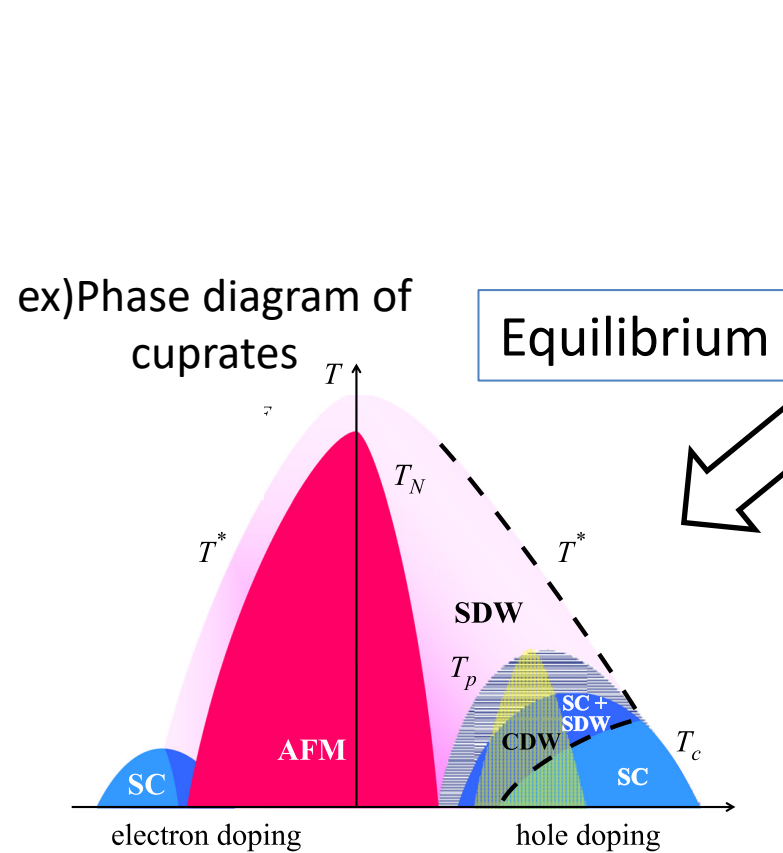


Potential impact on next generation photo-electronics technology & new spectroscopy techniques

ex) Fast memory, Spintronics, 6G telecommunication, Attosecond spectroscopy, etc..

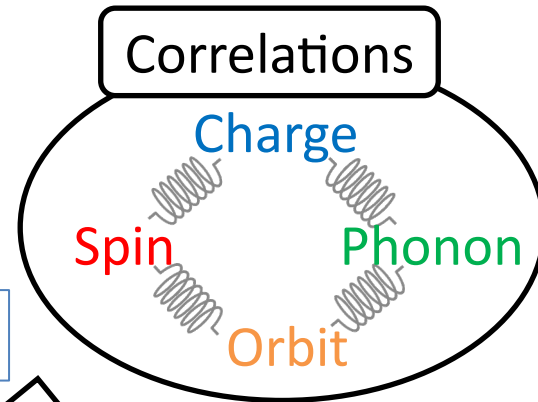
Strongly correlated systems: Crucial role of interactions between electrons

⇒ Various emergent collective phenomena in and out of equilibrium



**Rich physics & order**

A. Kordyuk, Low. Temp. Phys. **41**, 319 (2015)



Equilibrium

Noneq. • Nonlinear

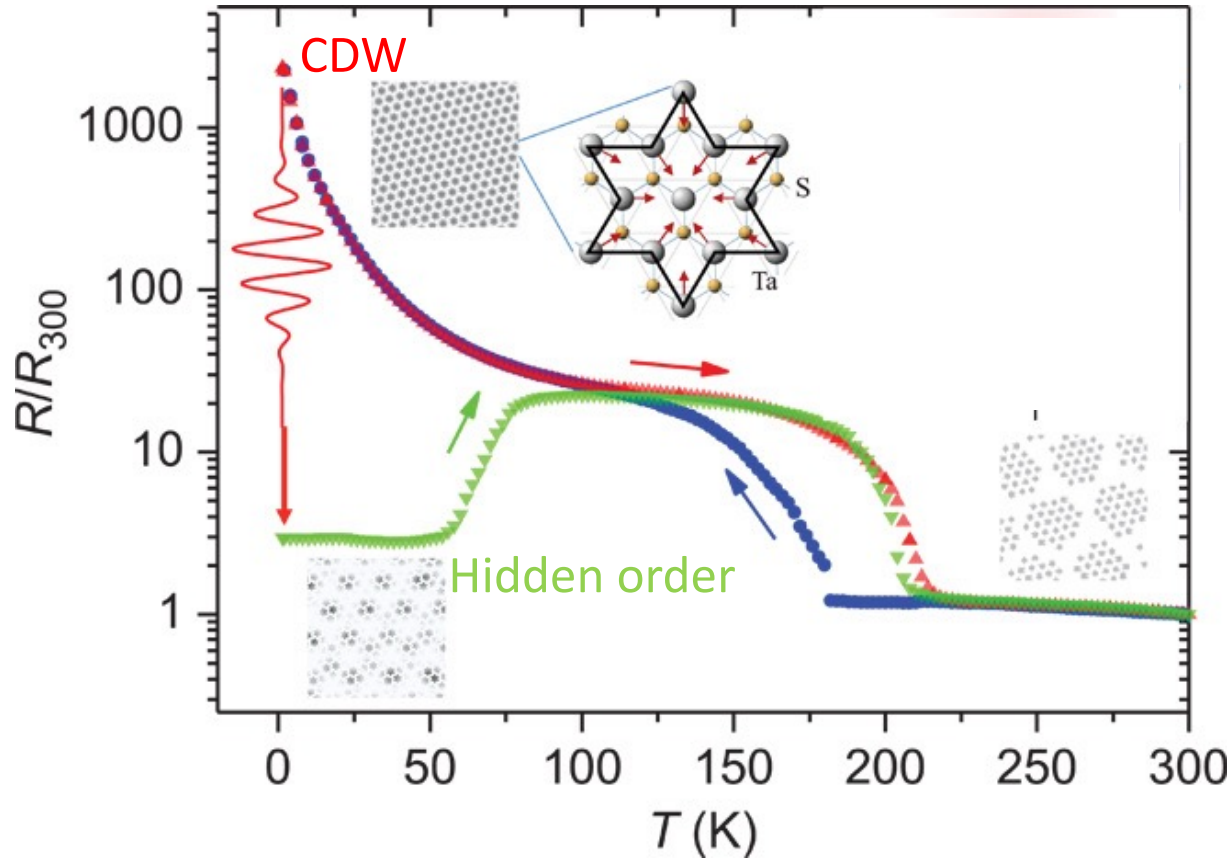
▷ Exotic excitation structure

→ Intriguing optical response

▷ Balance between different degrees of freedom

→ Rich photo-induced phase transitions

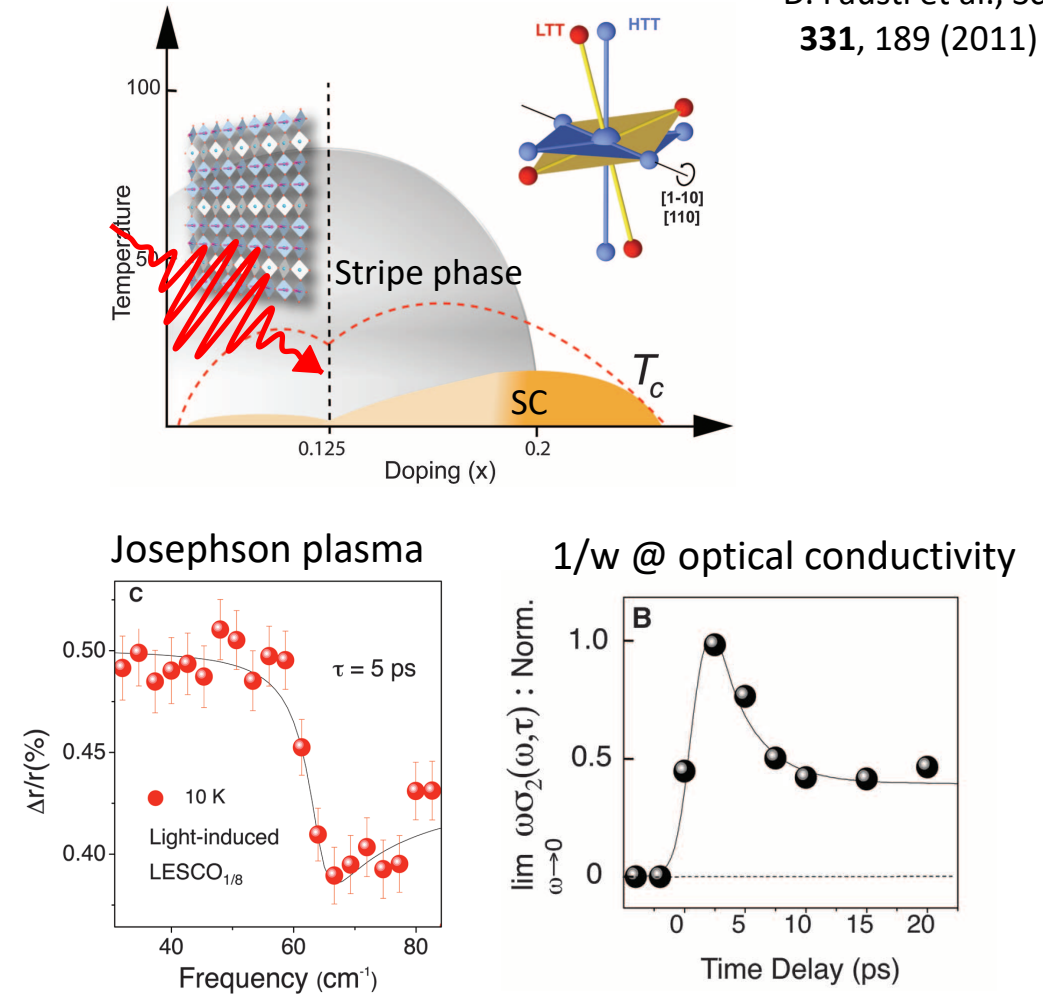
## Hidden CDW phase @ 1T-TaS<sub>2</sub>



Stojchevska et al., Science **344**, 6180 (2014);  
Vaskivskyi et al., Sci. Adv. **1**:e150016 (2015).

## Photoinduced SC @ La<sub>1.8-x</sub>Eu<sub>0.2</sub>Sr<sub>x</sub>CuO<sub>4</sub>

D. Fausti et al., Science **331**, 189 (2011).

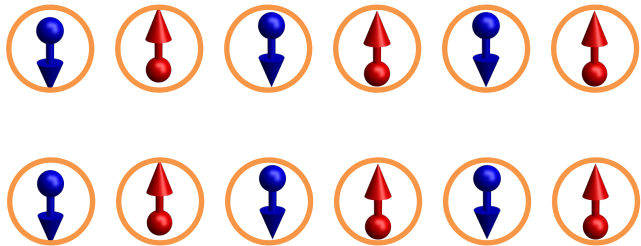


General question: Origin of nonequilibrium phases?

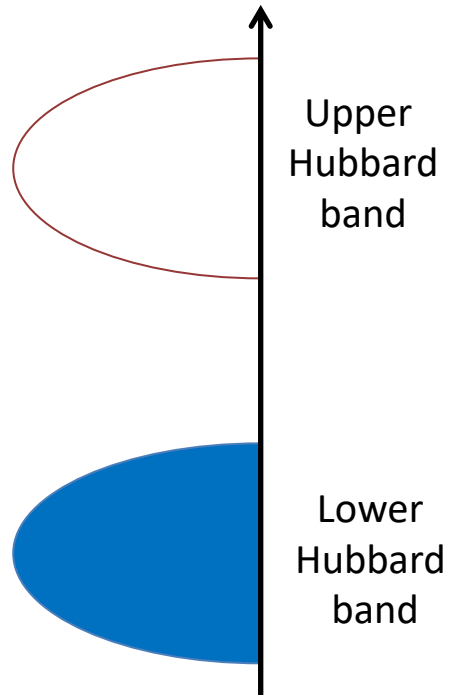
## Ex) Hubbard model

$$\hat{H} = -v \sum_{\langle i,j \rangle, \sigma} \hat{c}_i^\dagger \hat{c}_j + U \sum_i n_{i\uparrow} n_{i\downarrow}$$

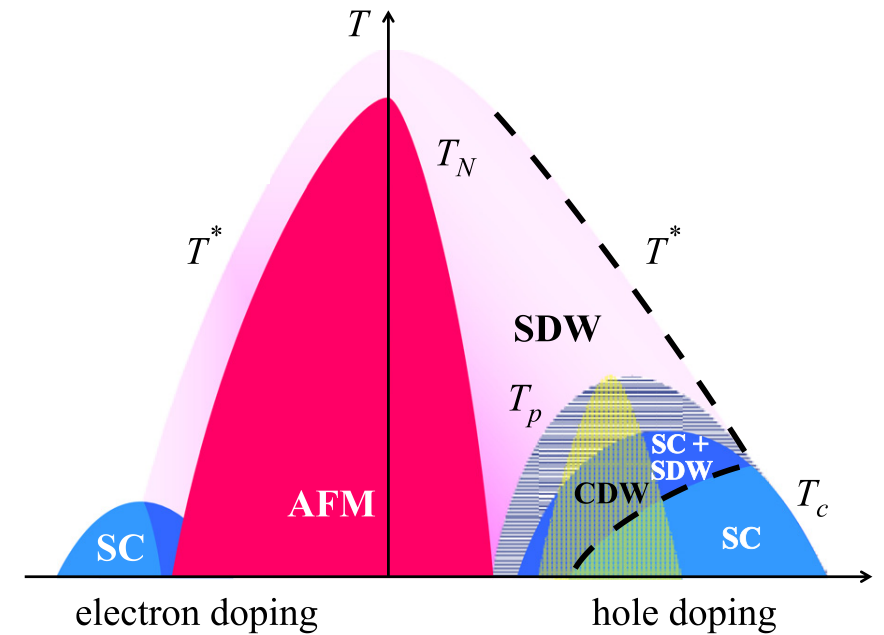
Mott Insulator @  $U \gg v$



Half filling: # electrons = # sites



## Phase diagram of cuprate



A. Kordyuk, Low. Temp. Phys. **41**, 319 (2015)

Doping activates correlations between spin, orbital and charge

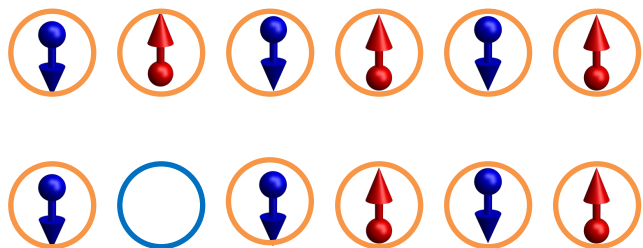


Emergence of rich phases

Ex) Hubbard model

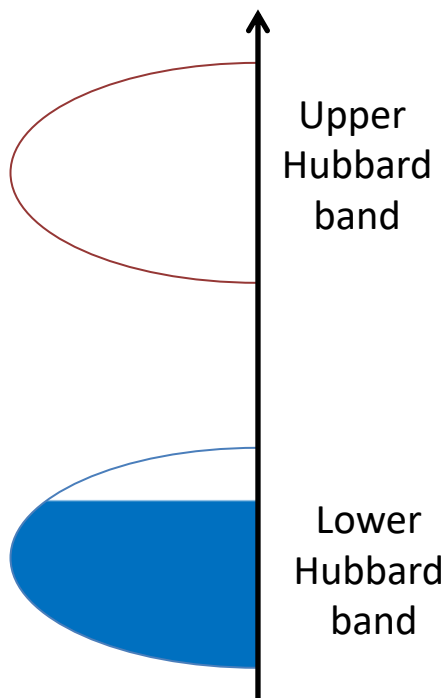
$$\hat{H} = -v \sum_{\langle i,j \rangle, \sigma} \hat{c}_i^\dagger \hat{c}_j + U \sum_i n_{i\uparrow} n_{i\downarrow}$$

Mott Insulator @  $U \gg v$

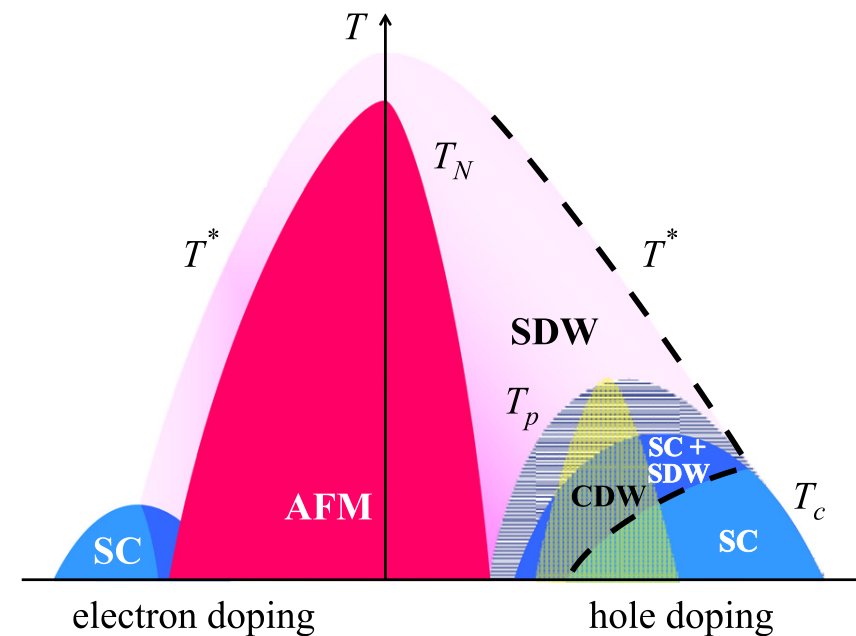


holon

ex) chemical doping



Phase diagram of cuprate



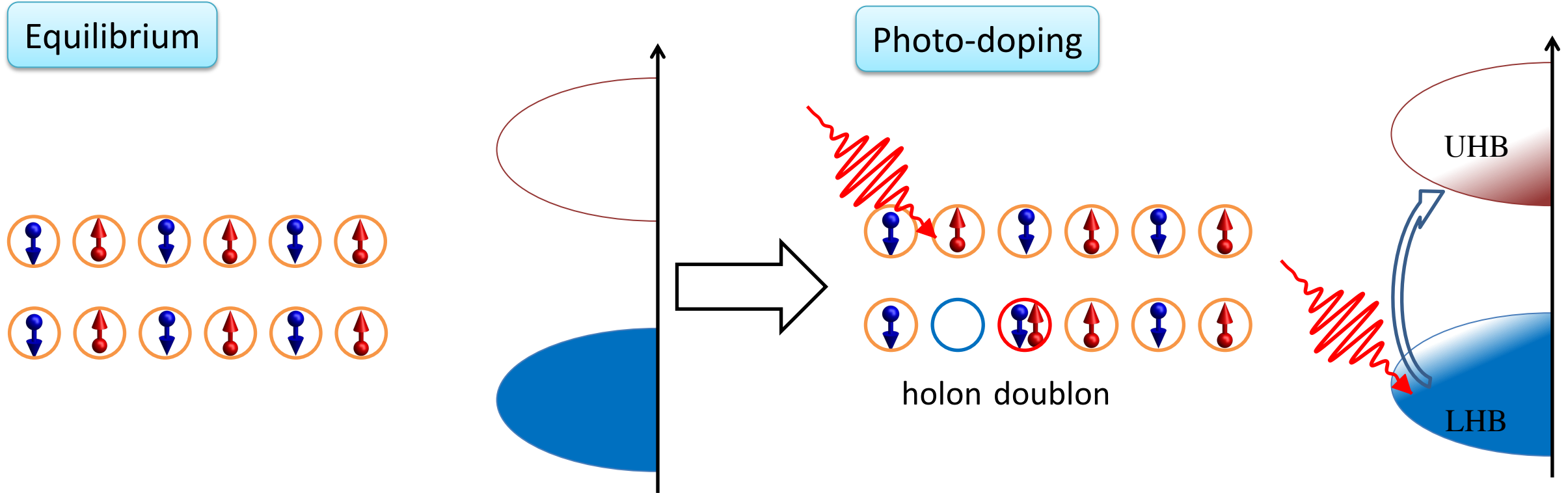
A. Kordyuk, Low. Temp. Phys. **41**, 319 (2015)

Doping activates correlations between spin, orbital and charge



Emergence of rich phases





**Various types of charge carriers are activated at the same time**

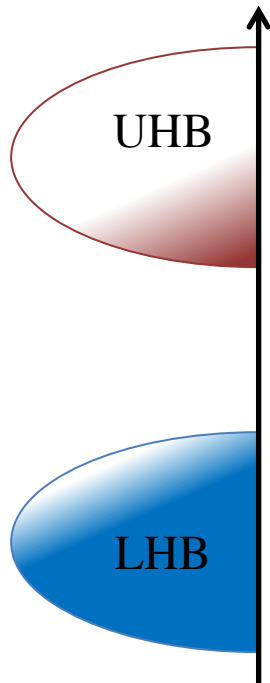
cf. Equilibrium doping  $\rightarrow$  holon **or** doublon

## Life-time of doublon • holon

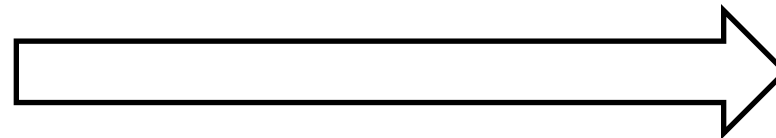
$$U \gg v \quad \Rightarrow \quad \tau_{\text{rec}} \gg 1/v \quad (\text{Exponential with } U/v)$$

N. Strohmaier, et. al., PRL **104**, 080401 (2010).  
R. Sensarma, et. al., PRB **82**, 224302 (2010).  
A. Rosch, et. al., PRL **101**, 265301 (2008).

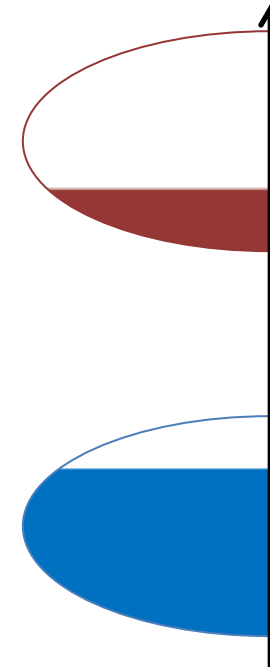
## Just after excitation



- (Approximate) conservation of doublons and holons
- Intraband relaxation + Cooling via environment



## Metastable steady state



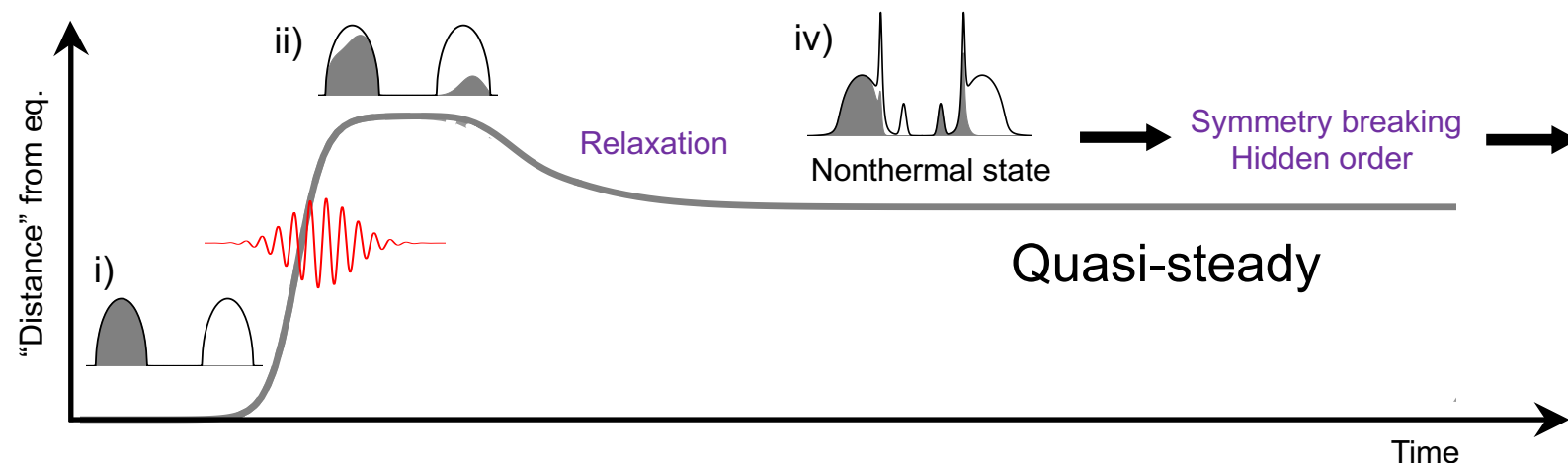
What kinds of metastable states emerge in photo-doped Mott insulators?

Review: YM, D Golež, M Eckstein, P Werner, arXiv:2310.05201

## 1) Direct time-evolution

Methods:

Exact Diagonalization,  
Tensor network,  
Dynamical mean-field theory,  
etc...

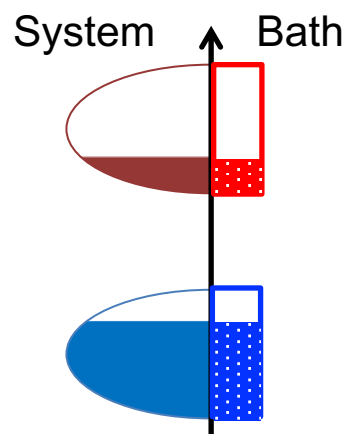


## 2) Quasi-NESS approach

Approximate quasi-steady state  
with a true steady state supported  
by external bath

J. Li, et. al., PRB **102**, 165136 (2020).

J. Li and M. Eckstein, PRB **103** 045133 (2021).



## 3) Quasi-equilibrium approach

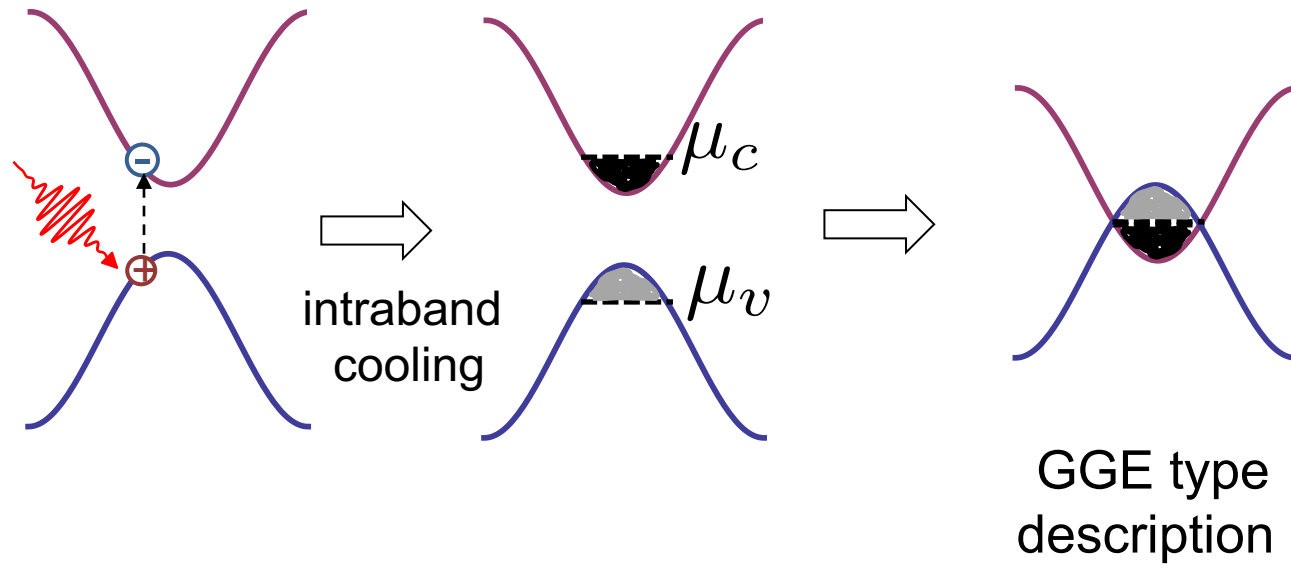
- ▷ Analogous to photo-doped semiconductor
- ▷ **Mainly used in this talk**

A. Rosch, et. al., PRL **101**, 265301 (2008).

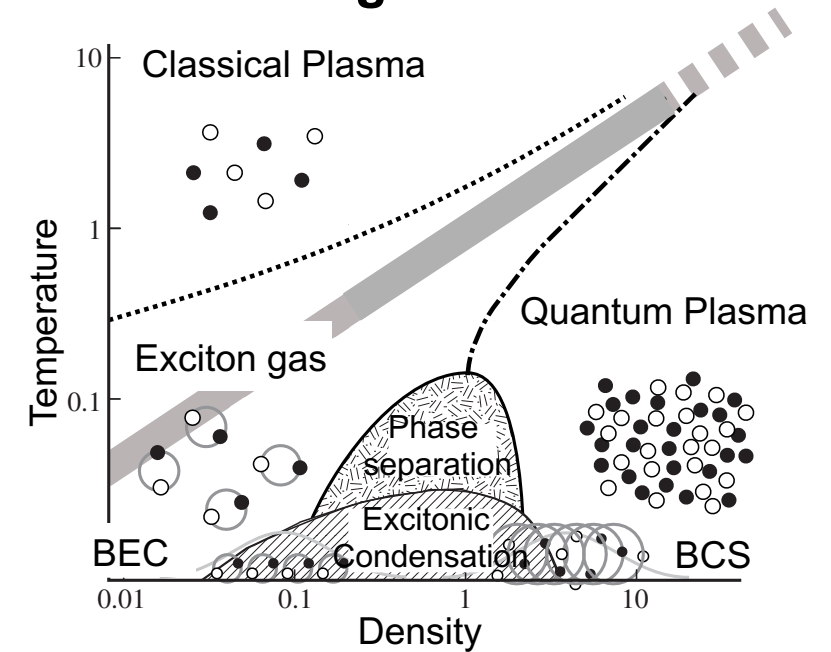
Y. Kanamori, et al., PRL **107**, 167403 (2011).

YM, et. al., Comm. Phys. **5**, 23 (2022).

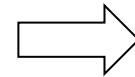
K. Asano, Bussei Kenkyu (2013).  
 L. V. Keldysh, *Contemporary Phys.* **27**, 395 (1986).



## Phase diagram



Conservation of electrons and holes



Effective equilibrium problem

Effective chemical potential & temperature  
 " $\mu_c, \mu_v, T_{\text{eff}}$ "

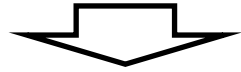
**Strongly correlated systems?**

Step1

Apply the Schrieffer-Wolff transformation (1/U expansion)

YM, et. al., Comm. Phys. **5**, 23 (2022).

Original Hamiltonian:  $\hat{H}$



Effective model with conserved local multiplets dressed with virtual fluctuation

Effective Hamiltonian:  $\hat{H}_{\text{eff}}$

ex) doublons, holons

Step2

Introducing **chemical potential** for local multiplets and effective temperature

$$\hat{K}_{\text{eff}} = \hat{H}_{\text{eff}} - \sum_{g \in \text{ps}} \mu_g \hat{n}_g \quad \hat{\rho}_{\text{eff}} = \exp(-\beta_{\text{eff}} \hat{K}_{\text{eff}}) \quad \text{GGE type description}$$

Step3

Solve the effective problem with existing **equilibrium** methods

$$\hat{H} = -v \sum_{\langle i,j \rangle, \sigma} \hat{c}_i^\dagger \hat{c}_j + \hat{H}_U + \hat{H}_V \quad \text{with} \quad \hat{H}_U = U \sum_i \hat{n}_{i\uparrow} \hat{n}_{i\downarrow} \quad U \gg v, V$$

$$\hat{H}_V = V \sum_{\langle ij \rangle} \hat{n}_i \hat{n}_j$$

Effective model with conserved local multiplets and effects of virtual fluctuation

$$\begin{aligned} \hat{H}_{\text{eff}} = & \hat{H}_U && \leftarrow \mathcal{O}(U) \\ & + \hat{H}_{\text{kin,LHB}} + \hat{H}_{\text{kin,UHB}} && \leftarrow \mathcal{O}(v) \quad \mathcal{O}(J_{\text{ex}}) \quad J_{\text{ex}} = \frac{4v^2}{U} \\ & + \hat{H}_{U,\text{shift}}^{(2)} + \hat{H}_{\text{spin,ex}} + \hat{H}_{\text{dh,ex}} + \hat{H}_{\text{kin,LHB}}^{(2)} + \hat{H}_{\text{kin,UHB}}^{(2)} + \hat{H}_{\text{dh,slide}}^{(2)} + \hat{H}_V && \leftarrow \mathcal{O}(J_{\text{ex}}) \end{aligned}$$

4 types of pseudo-particles

3 site terms

**Exchange coupling for spins**

$$\hat{H}_{\text{spin,ex}} = J_{\text{ex}} \sum_{\langle i,j \rangle} \hat{\mathbf{s}}_i \cdot \hat{\mathbf{s}}_j$$

**Exchange coupling for doublon-holon**

$$\hat{H}_{\text{dh,ex}} = -J_{\text{ex}} \sum_{\langle i,j \rangle} \hat{\eta}_i \cdot \hat{\eta}_j \quad \hat{\eta}_i^+ = (-)^i \hat{c}_{i\downarrow}^\dagger \hat{c}_{i\uparrow}^\dagger$$

$$\hat{\eta}_i^z = \frac{1}{2}(\hat{n}_i - 1)$$

Cold atom with extreme doping

A. Rosch, et. al., PRL **101**, 265301 (2008).

Metastable state with doublon or holon

$$\hat{H}_{\text{dh,ex}} = -J_{\text{ex}} \sum_{\langle i,j \rangle} \hat{\boldsymbol{\eta}}_i \cdot \hat{\boldsymbol{\eta}}_j \quad \hat{\eta}_i^+ = (-)^i \hat{c}_{i,\downarrow}^\dagger \hat{c}_{i,\uparrow}^\dagger$$

$$\hat{\eta}_i^z = \frac{1}{2}(\hat{n}_i - 1) \quad \Rightarrow \quad |\Psi\rangle = e^{-i\theta \sum_i S_i^x} |\uparrow\uparrow\uparrow \dots\rangle = e^{-i\frac{\theta}{2} \sum_i (-1)^i (c_{i\uparrow}^\dagger c_{i\downarrow}^\dagger + \text{H.c.})} |0\rangle,$$

$$\cos\theta = 1 - 2n_d \quad \langle c_{i\uparrow}^\dagger c_{i\downarrow}^\dagger \rangle = \frac{(-1)^i}{2} \sin\theta$$

$\text{SU}_c(2)$  Symmetry

**$\eta$  pairing state**

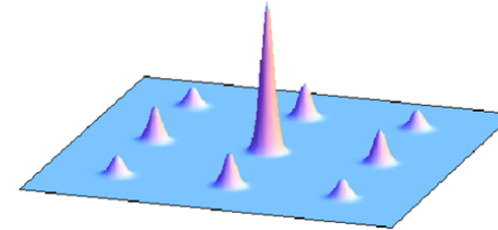
Yang's  $\eta$ -pairing state

$$|\phi_{N_\eta}\rangle = \frac{1}{\sqrt{\mathcal{C}_{N_\eta}}} (\hat{\eta}^+)^{N_\eta} |0\rangle$$

$$\hat{\eta}^+ = \sum_j (-1)^j \hat{c}_{j,\downarrow}^\dagger \hat{c}_{j,\uparrow}^\dagger = \sum_k \hat{c}_{\underline{\pi-k},\downarrow}^\dagger \hat{c}_{\underline{k},\uparrow}^\dagger$$

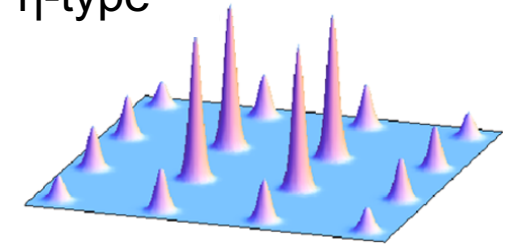
C. N. Yang, Phys. Rev. Lett. 63, 2144 (1989)

normal s-wave



$\mathbf{k} = (0,0)$

$\eta$ -type

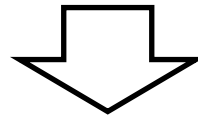


$\mathbf{k} = (\pi,\pi)$

Momentum distribution of fermion pair

## Quasi-equilibrium approach for 1D extended Hubbard model

- ▶ Numerical analysis (A tensor network: iTEBD) [YM, et al., Comm. Phys. 5, 23 \(2022\).](#)
- ▶ Analytical discussion [YM, et al., Phys. Rev. Lett. 130, 106501 \(2023\).](#)



## Main points

- ▶ Exact form of wave function of photo-doped states:  $|\Psi\rangle = |\Psi_{SF}^{GS}\rangle |\Psi_{spin}^{GS}\rangle |\Psi_{\eta-spin}^{GS}\rangle$
- ▶ Spin, charge and  $\eta$ -spin separation
- ▶ Intuitive insight into physics of metastable states

**Emergent degrees of freedoms by photo-doping lead to intriguing nonequilibrium phases!**



Wave function @  $U \rightarrow \infty$ ,  $V/J_{ex} = \text{const}$ ,  $T_{eff} = 0$

$$|\Psi\rangle = \underbrace{|\Psi_{SF}^{GS}\rangle}_{\substack{\text{Spinless fermion} \\ \text{(Position of Singlons)} \\ H_{SF, free}}} \underbrace{|\Psi_{spin}^{GS}\rangle}_{\substack{\text{Squeezed} \\ \text{spin space} \\ H_{spin}^{(SQ)}}} \underbrace{|\Psi_{\eta\text{-spin}}^{GS}\rangle}_{\substack{\text{Squeezed} \\ \eta\text{-spin space} \\ H_{\eta\text{-spin}}^{(SQ)}}$$

- ▷ Extension of Ogata-Shiba state in equilibrium  $|\Psi\rangle = |\Psi_{SF}^{GS}\rangle |\Psi_{spin}^{GS}\rangle$
- ▷ Spin, charge and  $\eta$ -spin separation
- ▷ Useful insight into physics

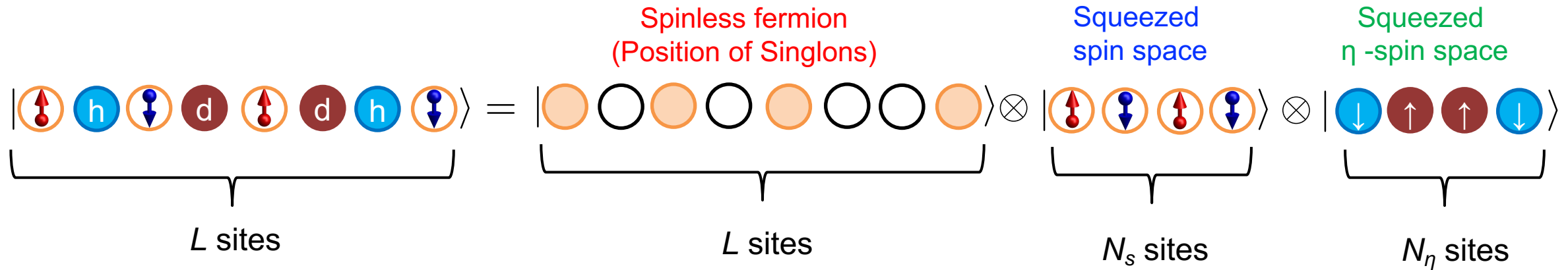
M. Ogata & H. Shiba,  
PRB 41 2326 (1990).

# Explanation of $|\Psi\rangle = |\Psi_{SF}^{GS}\rangle |\Psi_{\sigma}^{GS}\rangle |\Psi_{\eta}^{GS}\rangle$

YM, et al., PRL. **130**, 106501 (2023).

New expression of states:  $\hat{U}$

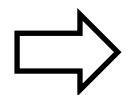
$L$ : System size  
 $N_s$ : Number of singly occupied sites  
 $N_{\eta}$ : Number of doublons and holons



Hamiltonian for  $J_{ex} = 0$  in the new expression

0 th order wave function

$$\hat{U} \hat{H}_{kin} \hat{U}^{\dagger} = -t_{hop} \sum_{\langle i,j \rangle} (\hat{c}_i^{\dagger} \hat{c}_j + h.c.) (\equiv \hat{H}_{SF,free})$$



$$|\Psi_{SF}^{GS}\rangle |\Psi_{\sigma,\eta}\rangle$$

i.e. Degeneracy of  $2^{N_s} \cdot 2^{N_{\eta}}$

$|\Psi_{\sigma,\eta}\rangle$  is determined by degenerate perturbation theory

YM, et al., PRL. **130**, 106501 (2023).

$\mathcal{O}(J_{\text{ex}})$  terms projected to  $|\Psi_{\text{SF}}^{\text{GS}}\rangle |\sigma\rangle |\eta\rangle$

$$\hat{H}_{\text{spin}}^{(\text{SQ})} = J_{\text{ex}}^s \sum_i \hat{\mathbf{s}}_{i+1} \cdot \hat{\mathbf{s}}_i,$$

$$\hat{H}_{\eta\text{-spin}}^{(\text{SQ})} = -J_X^\eta \sum_j (\hat{\eta}_{j+1}^x \hat{\eta}_j^x + \hat{\eta}_{j+1}^y \hat{\eta}_j^y) + J_Z^\eta \sum_j \hat{\eta}_{j+1}^z \hat{\eta}_j^z,$$

$$J_{\text{ex}}^s = (\tilde{x} - \tilde{x}') J_{\text{ex}}$$

$$J_X^\eta = (\tilde{y} - \tilde{y}') J_{\text{ex}}$$

$$J_Z^\eta = -(\tilde{y} - \tilde{y}') J_{\text{ex}} + 4\tilde{y}V$$

with

## 2-site terms

$$\tilde{x} = n_s - \frac{\sin^2(\pi n_s)}{\pi^2 n_s},$$

$$\tilde{y} = n_\eta - \frac{\sin^2(\pi n_\eta)}{\pi^2 n_\eta},$$

## 3-site terms

$$\tilde{x}' = \frac{\sin(2\pi n_s)}{2\pi} - \frac{\sin^2(\pi n_s)}{\pi^2 n_s},$$

$$\tilde{y}' = \frac{\sin(2\pi n_\eta)}{2\pi} - \frac{\sin^2(\pi n_\eta)}{\pi^2 n_\eta}.$$

$n_s$ : Density of singly occupied sites

$n_\eta$ : Density of doublons and holons

- ▷ spin and  $\eta$ -spin are separated
- ▷ Exchange couplings are renormalized

## Summary

$$|\Psi\rangle = \underbrace{|\Psi_{\text{SF}}^{\text{GS}}\rangle}_{H_{\text{SF},\text{free}}} \underbrace{|\Psi_{\text{spin}}^{\text{GS}}\rangle}_{H_{\text{spin}}^{(\text{SQ})}} \underbrace{|\Psi_{\eta\text{-spin}}^{\text{GS}}\rangle}_{H_{\eta\text{-spin}}^{(\text{SQ})}}$$

YM, et al., PRL. **130**, 106501 (2023).

$\eta$ -spin sectors

Described by the XXZ model



Two types of phases

$J_z < J_x$  : **Gapless** phase of the XXZ model

$J_z > J_x$  : **Gapful** phase of the XXZ model

$\eta$ -pairing state with slowly decaying

CDW state with slowly decaying

$$\chi_{\text{pair}}(r) \equiv \langle \hat{\eta}^x(r) \hat{\eta}^x(0) \rangle$$

$$\chi_{\text{charge}}(r) \equiv \langle \hat{\eta}^z(r) \hat{\eta}^z(0) \rangle$$

⌘ Alternating sign in definition of  $\hat{\eta}_i^+ = (-)^i \hat{c}_{i\downarrow}^\dagger \hat{c}_{i\uparrow}^\dagger$

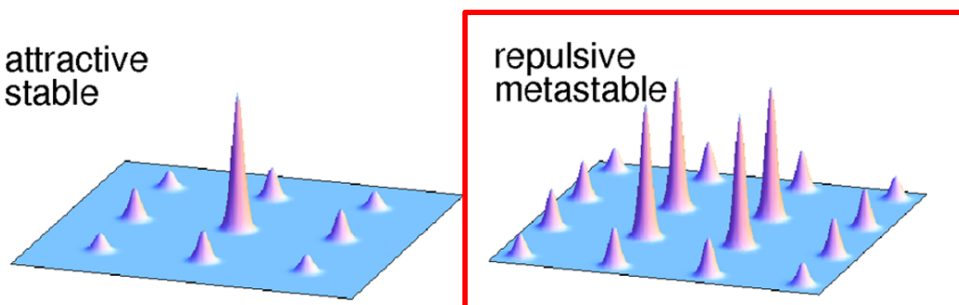
⌘ Long range order in the squeezed  $\eta$  spin space

⇒ usual pair correlations are staggered

⇒ **String type order** cf. Haldane phase

attractive  
stable

repulsive  
metastable

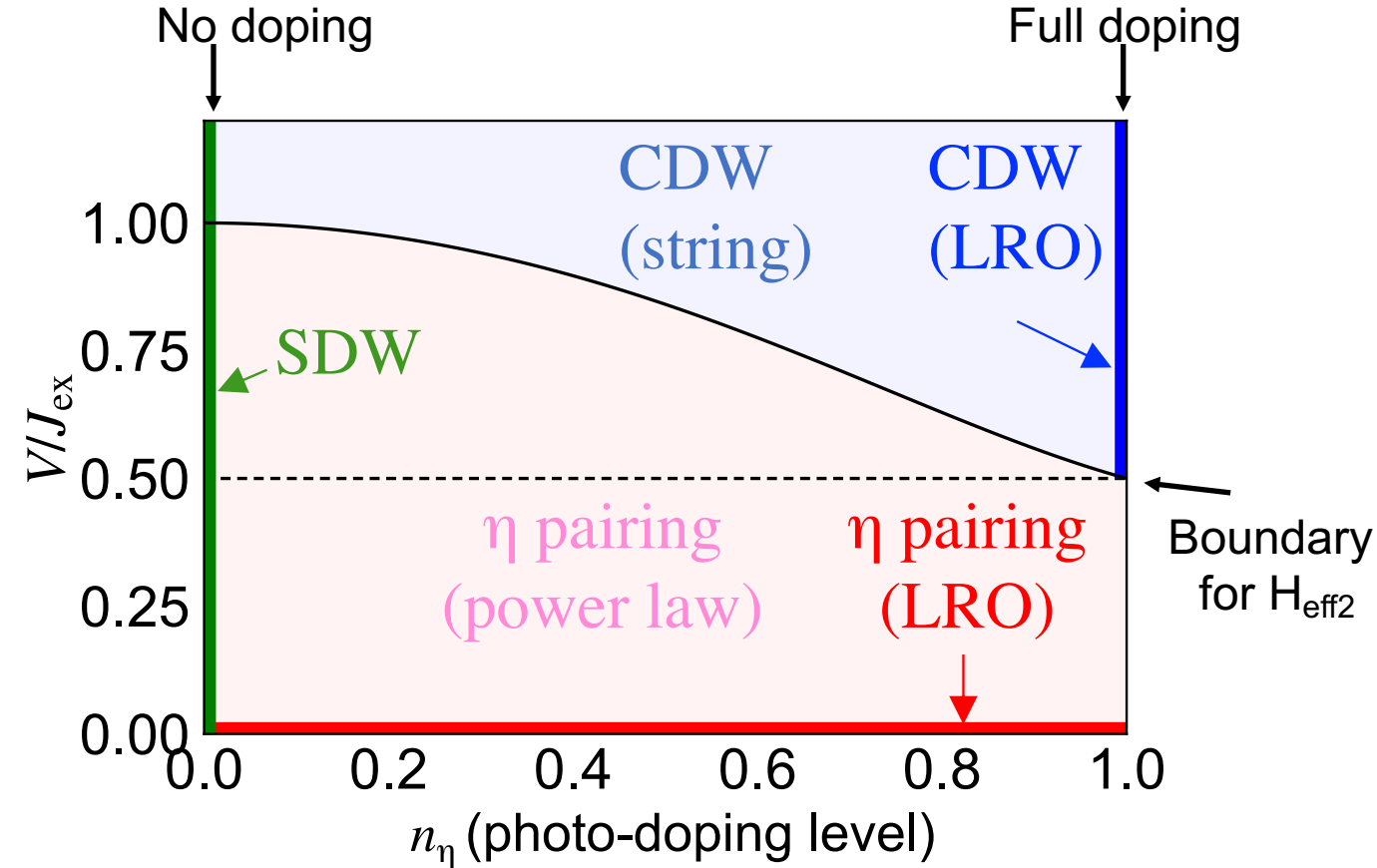


A. Rosch, et. al., PRL **101**, 265301 (2008).

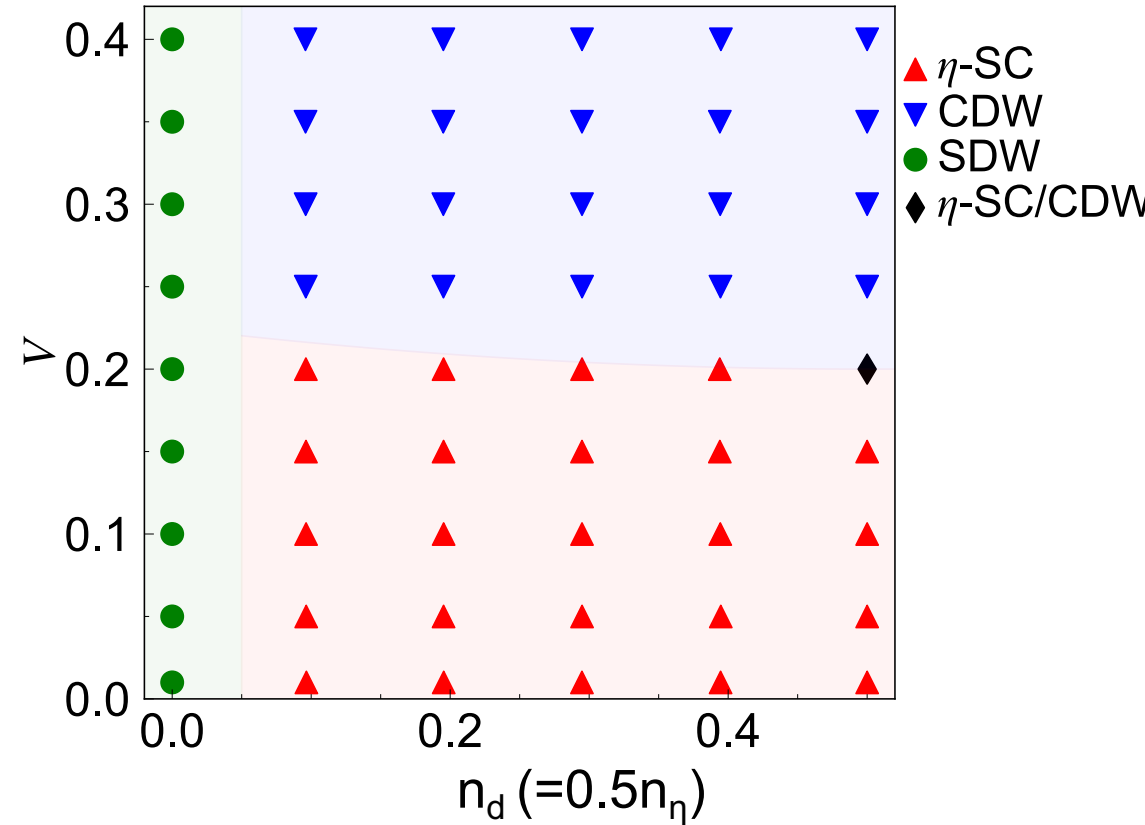


# Phase diagram of the photo-doped states at $T_{\text{eff}} = 0$

$U \rightarrow \infty$  phase diagram @ half-filling



iTEBD results for  $H_{\text{eff}2}$  with  $J_{\text{ex}} = 0.4$



- ▷ 3 site terms favor  $\eta$  pairing phase
- ▷ Analytic argument well explains numerically obtained phase diagram for  $H_{\text{eff}2}$  (no 3 site terms)
- ▷ Picture at  $U \rightarrow \infty$  works well even for finite  $U$

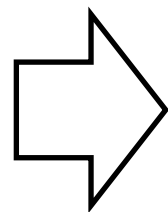
Total central charge ( $c$ )  $\sim$  Number of massless modes

$$|\Psi\rangle = \underbrace{|\Psi_{SF}^{GS}\rangle}_{H_{SF,free}} \underbrace{|\Psi_{spin}^{GS}\rangle}_{H_{spin}^{(SQ)}} \underbrace{|\Psi_{\eta-spin}^{GS}\rangle}_{H_{\eta-spin}^{(SQ)}}$$

▷ **Charge (SF) sector:** gapless

▷ **Spin sector:** gapless

▷  **$\eta$ -spin sector:**  $\begin{cases} \eta \text{ pairing} \rightarrow \text{gapless} \\ \text{CDW} \rightarrow \text{gapful} \end{cases}$



$\eta$  pairing:  $c=3$  ? & CDW:  $c=2$  ?

## Scaling analysis

J. A. Kjäll, et al., PRB 87, 235106 (2013).

$$S_E = \frac{c}{6} \ln(\xi_D) + s_0$$

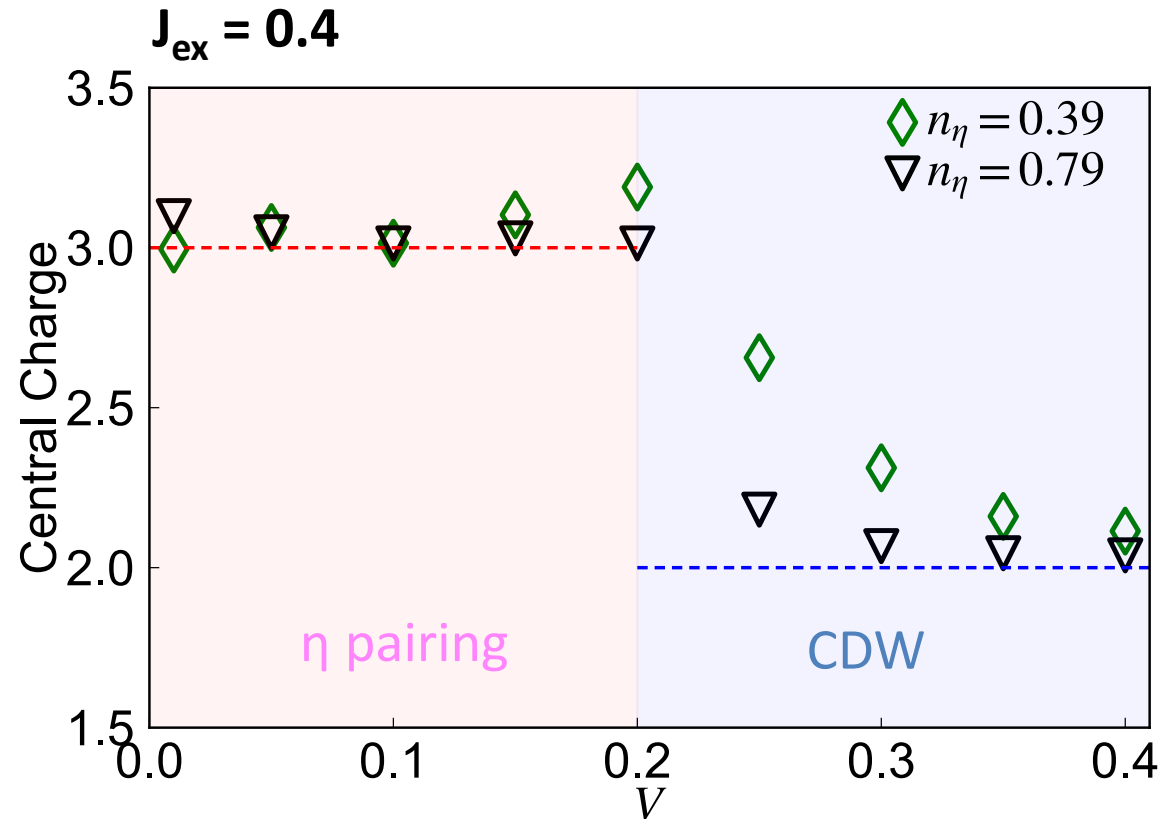
$c$ : central charge

$D$ : cut-off dimension

$\xi_D$ : correlation length at  $D$

$S_E$ : entanglement entropy at  $D$

$\eta$  pairing:  $c=3$  & CDW:  $c=2$



$c=3$  in single-band Hubbard model is not expected in equilibrium



Emergence of extra degrees of freedom by photo-doping!

$$A_k(\omega) = -\frac{1}{\pi} \text{Im} G_k^R(\omega) \quad \text{with} \quad G_k(t, t') = -i \langle \mathcal{T} c_k(t) c_k^\dagger(t') \rangle$$

Equilibrium doped system

**Electron** = charge (SF) degree + spin degree  
gapless                      gapless

⇒ **Gapless** around Fermi level

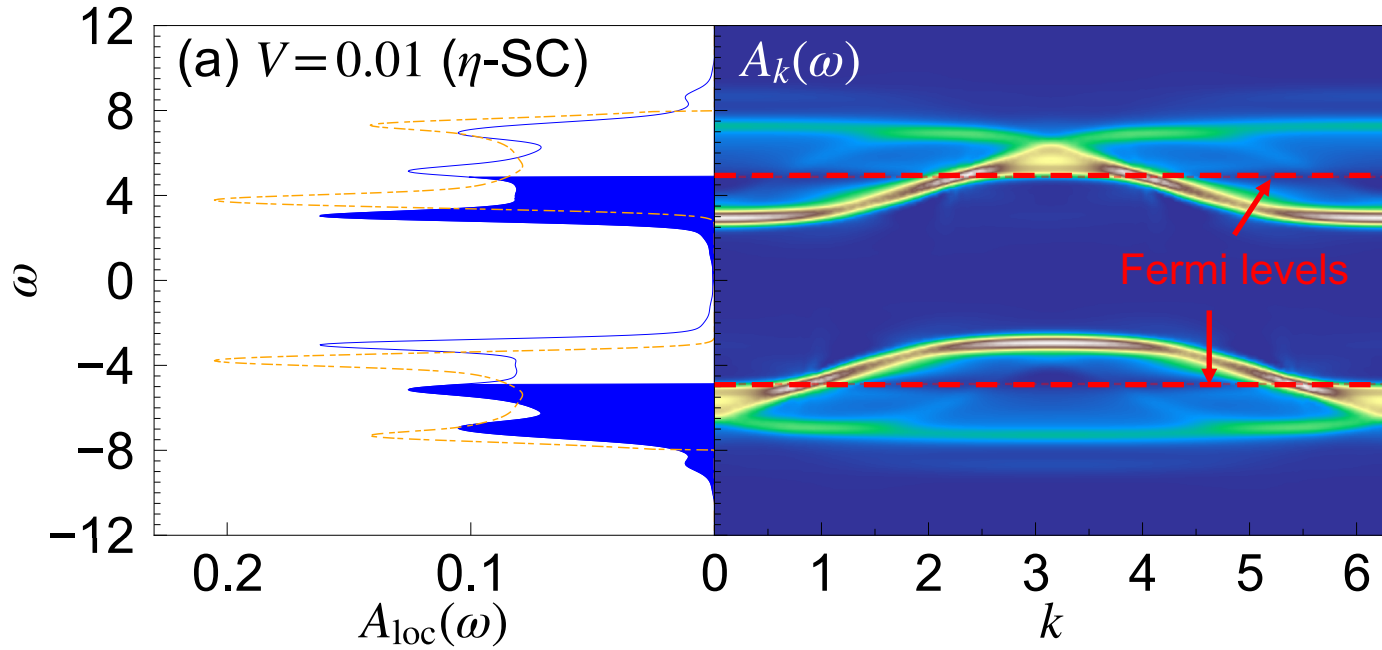
Photo-doped system

**Electron** = charge (SF) degree + spin degree +  $\eta$  spin degree  
gapless                      gapless                       $\eta$  pairing: gapless  
CDW: gapful

⇒  $\eta$  pairing phase : **Gapless** around Fermi level ?  
CDW phase : **Gapful** around Fermi level ?



# Single particle spectra for $\eta$ pairing state and CDW state

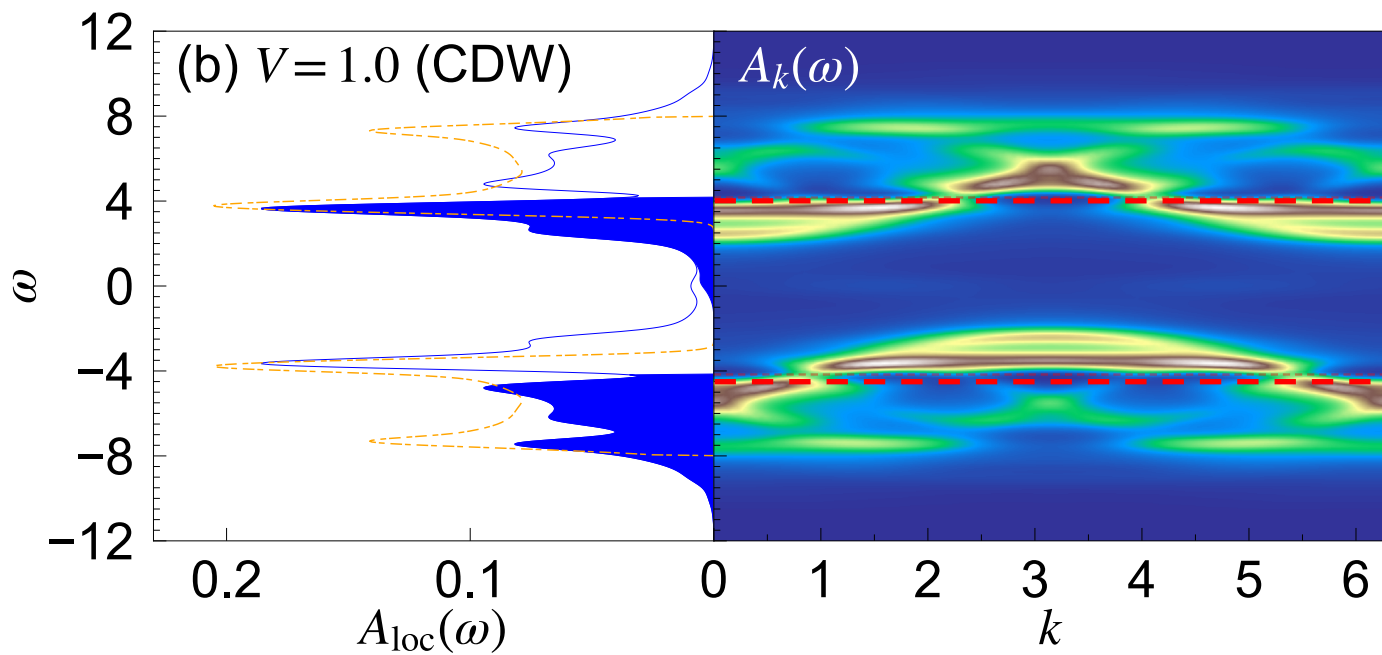


$\eta$  pairing state

No gap close to the Fermi level

⇒ Gapless SC

cf. DMFT J. Li et al, Mod. Phys. Lett B (2022)



CDW

Finite Gap at the Fermi level

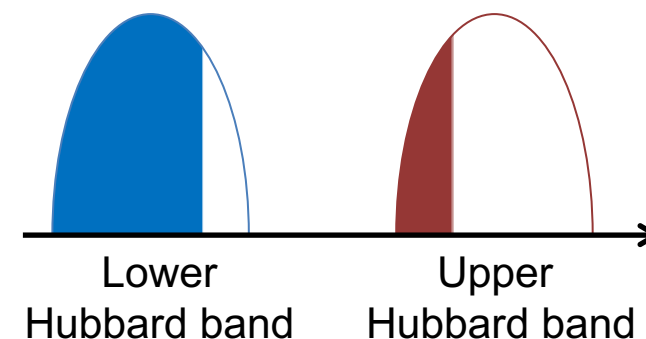
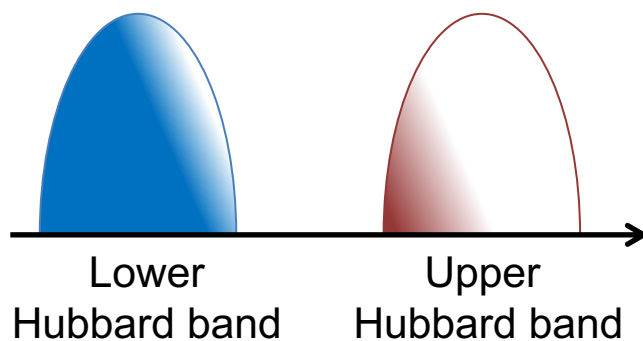
Large gap Mott system



Approximate conservation of charge carriers



Quasi steady states



## Photo-doped states in 1D extended Hubbard model

- ▶ Extension of Ogata-Shiba state in equilibrium :  $|\Psi\rangle = |\Psi_{SF}^{GS}\rangle |\Psi_{spin}^{GS}\rangle |\Psi_{\eta-spin}^{GS}\rangle$
- ▶ Spin, charge and  $\eta$ -spin separation
- ▶ Intuitive insight into physics of metastable states

YM, et al., Comm. Phys. **5**, 23 (2022):

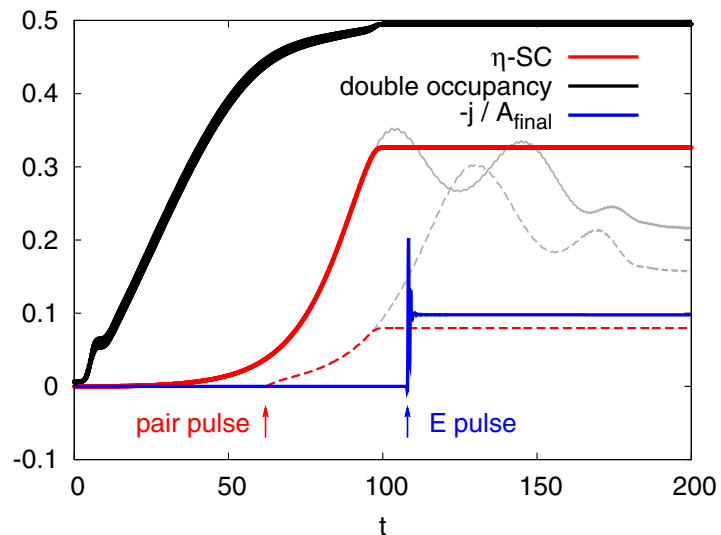
YM, et al., PRL. **130**, 106501 (2023).

**Emergent degrees of freedoms by photo-doping lead to intriguing nonequilibrium phases!**

# Supplement

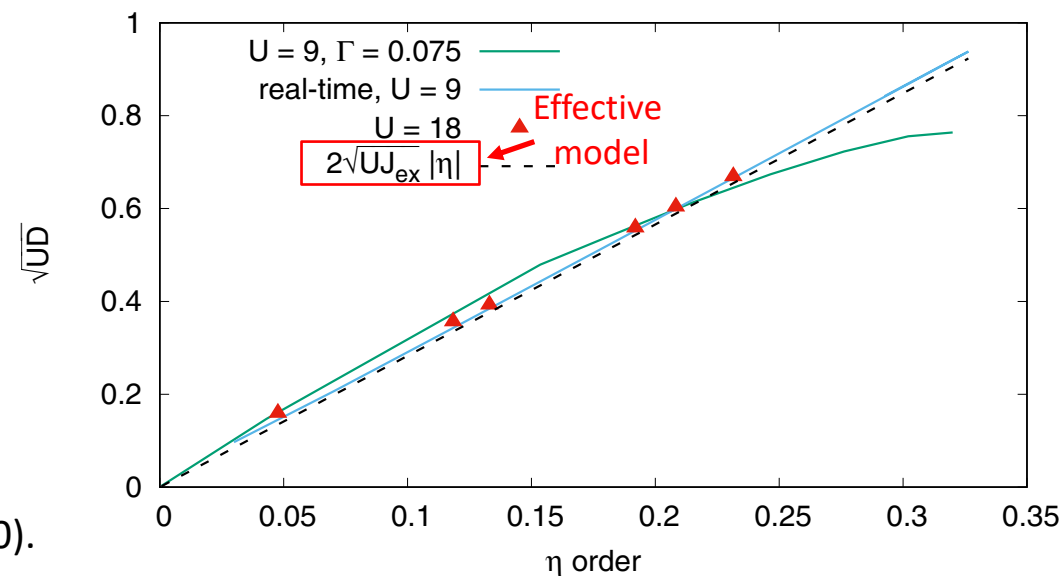
## DMFT time evolution

P. Werner, et al., PRB 100, 155130 (2019).



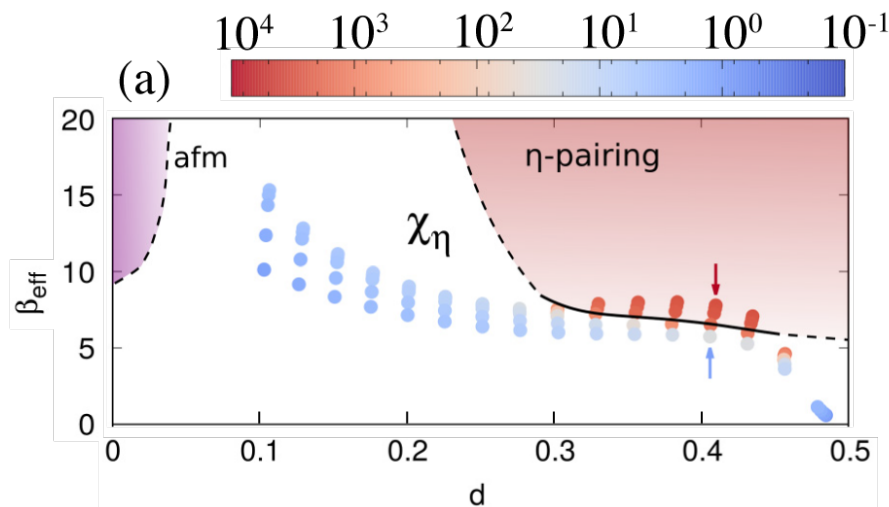
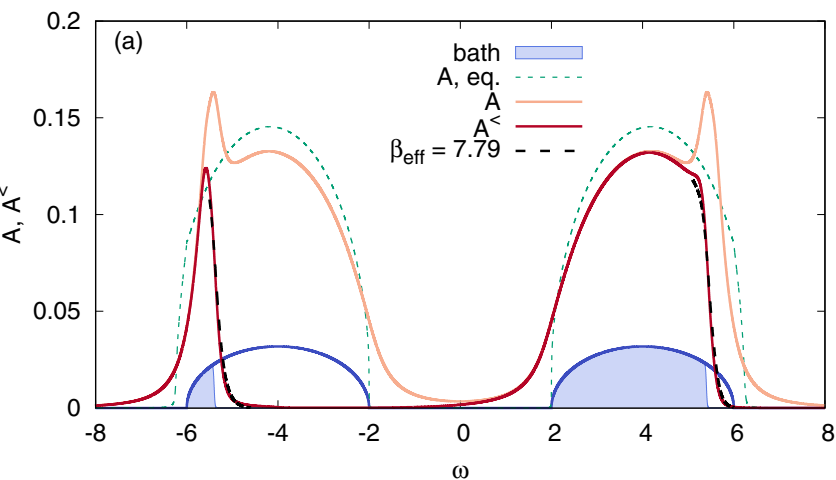
## Comparison of three approaches

D: superfluid density



## Ness approach with DMFT

J. Li, et. al., PRB 102, 165136 (2020).



▷ Three approaches are consistent

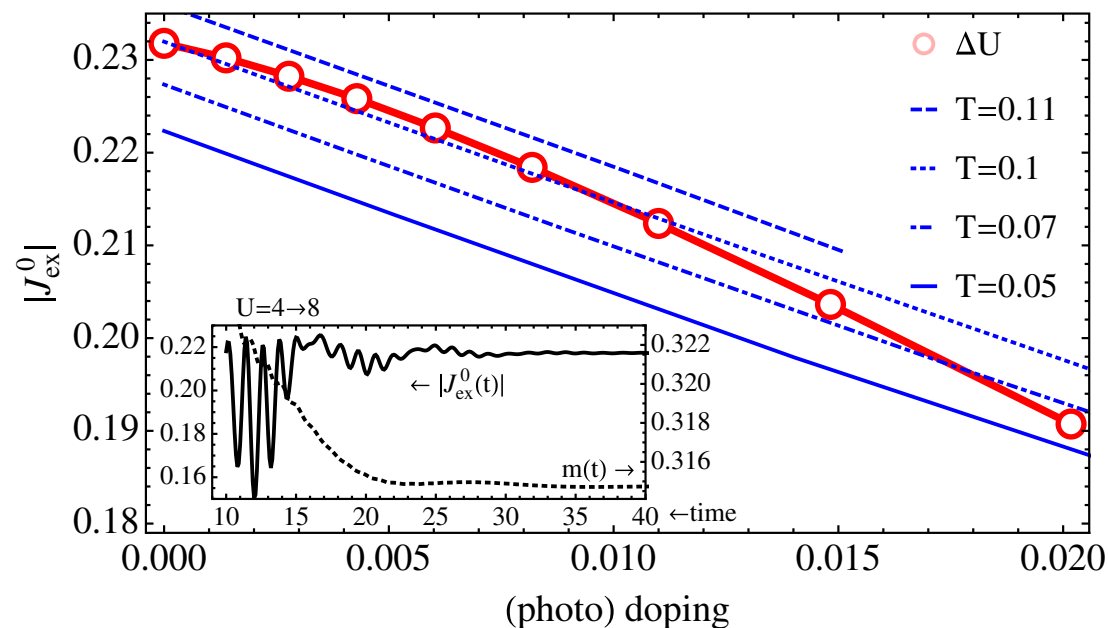
# Indication to spin properties

$$|\Psi\rangle = \underbrace{|\Psi_{SF}^{GS}\rangle}_{H_{SF,free}} \underbrace{|\Psi_{spin}^{GS}\rangle}_{H_{spin}^{(SQ)}} \underbrace{|\Psi_{\eta-spin}^{GS}\rangle}_{H_{\eta-spin}^{(SQ)}}$$

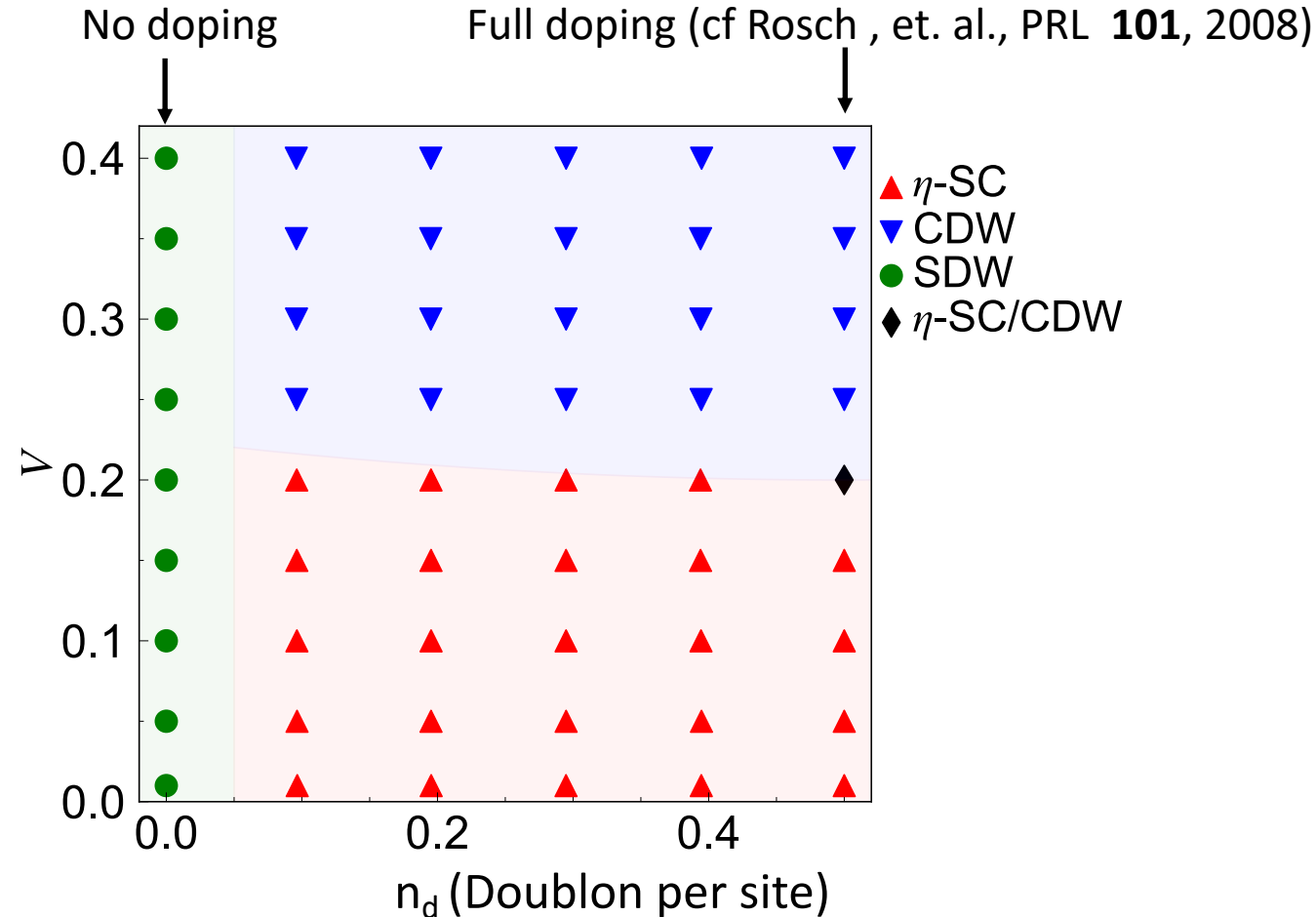
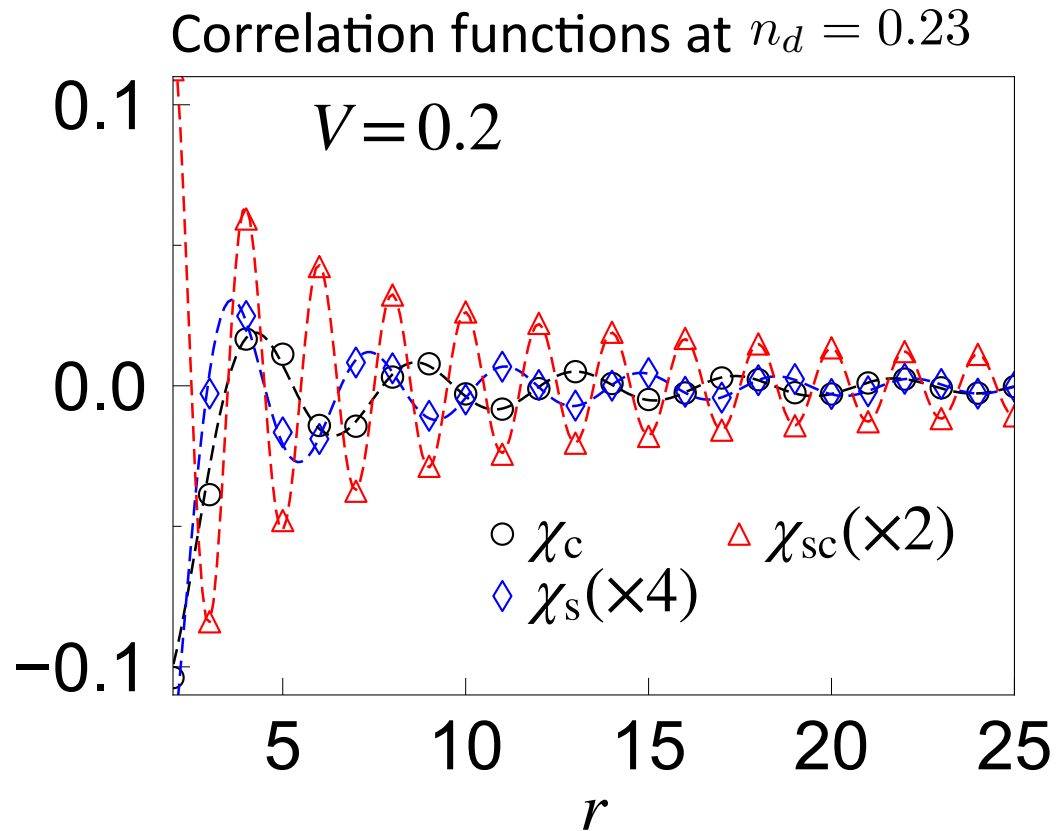
SF and spin part is independent of the ratio between  $N_h$  and  $N_d$ .

⇒ Chemical doping and photo-doping have the same effect on spin correlations

cf. DMFT results



J. Mentink & M. Eckstein  
PRL 113 057301 (2014).



- ▷ Quasi-long range order

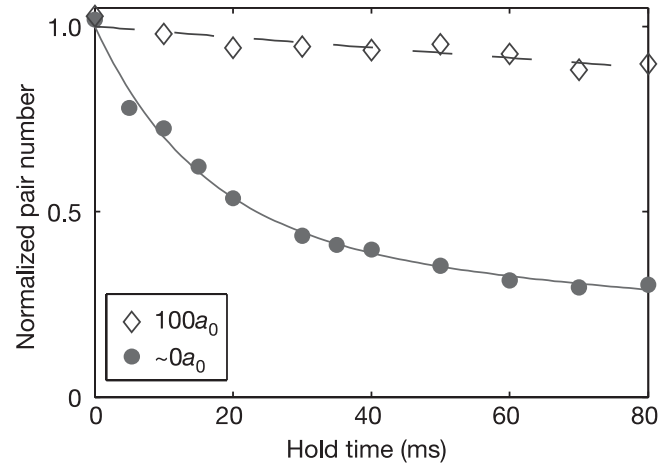
$$\chi(r) \propto \cos(qr)/r^a \quad \text{with} \quad q = \pi \text{ } (\eta\text{-SC}) \quad q = 2n_d\pi \text{ (CDW)} \quad q = (1 - 2n_d)\pi \text{ (SDW)}$$

- ▷ Boundary of  $\eta$ SC and CDW  $\hat{=} V=J_{ex}/2$

⇒ Special kinematics of doublons and holons in one dimensional system

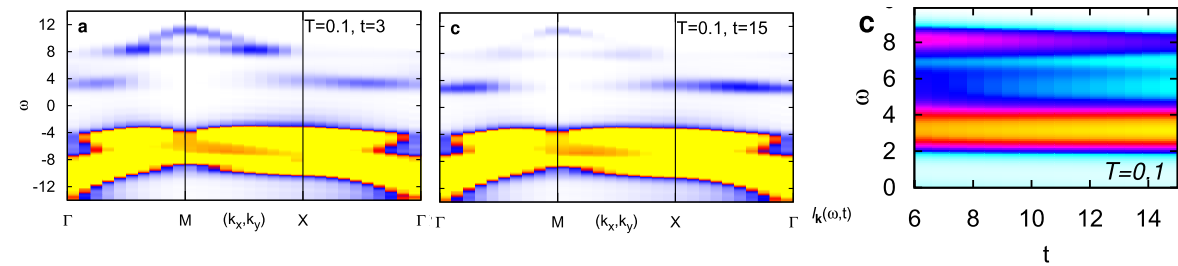
## Cold Atoms

K. Winkler et al., Nature **441**, 853 (2006).



## Cooling of carriers in Photo-doped Mott

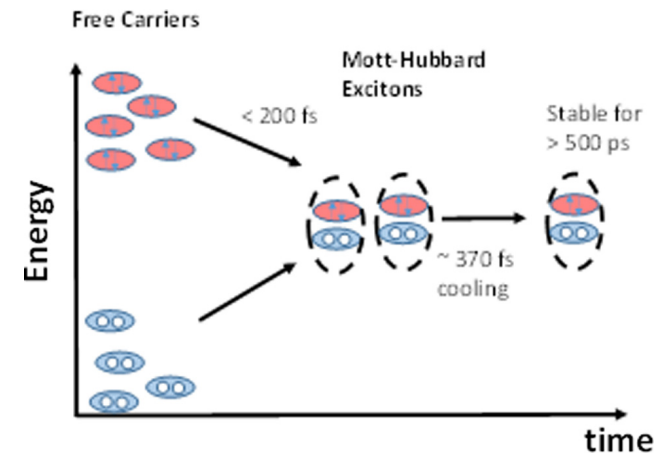
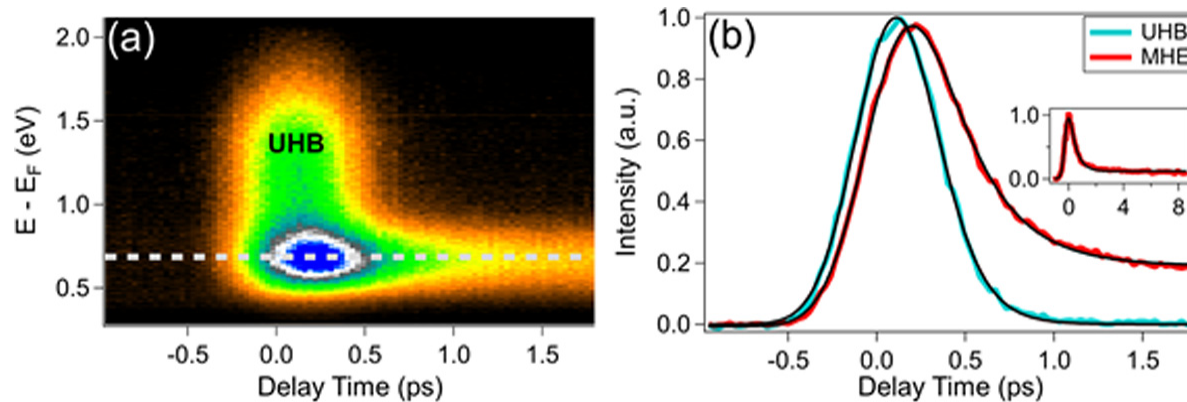
### Cluster DMFT study



M. Eckstein & P. Werner Sci. Rep. **6** 21235 (2015)

## $\alpha$ -RuCl<sub>3</sub>

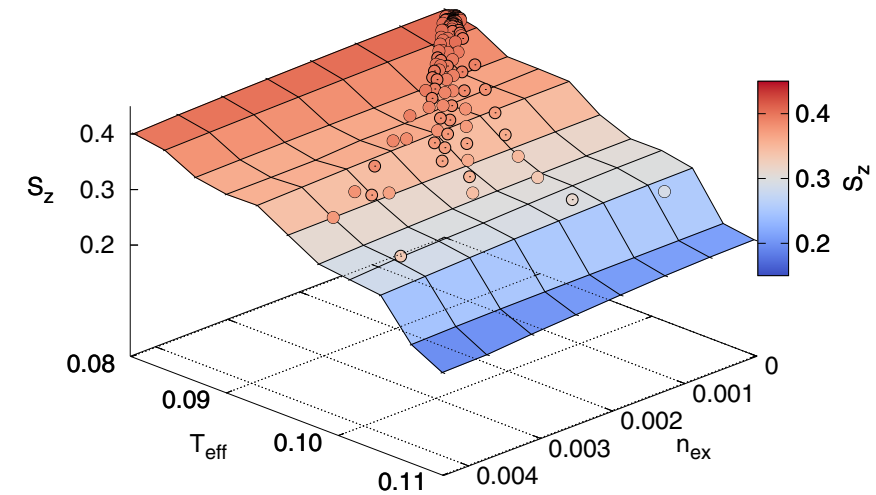
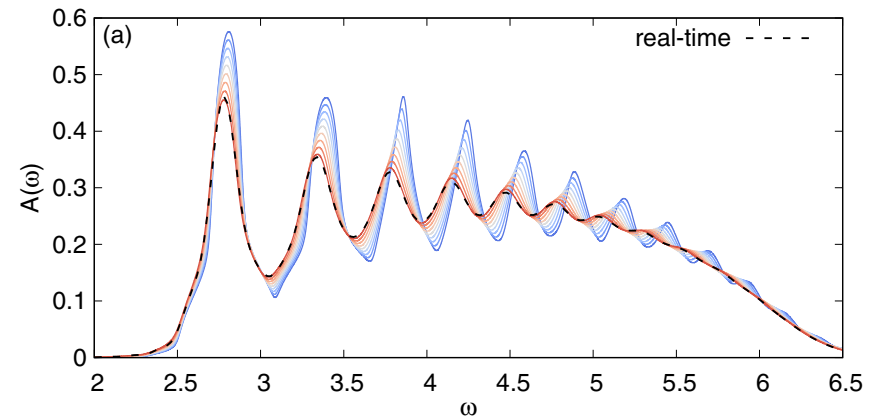
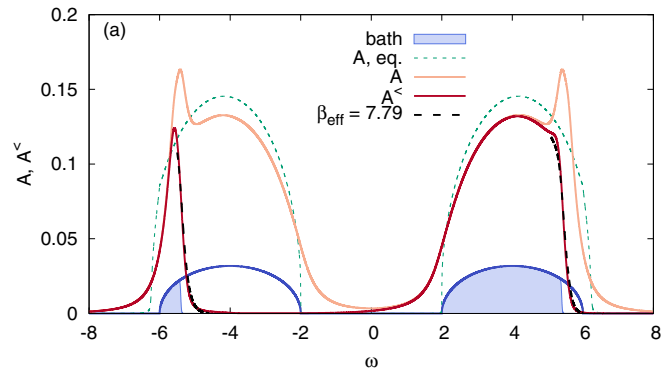
D. Nevola, et al., PRB **103**, 245105 (2021)



## NESS @ coupling with heat and particle bath

J. Li, et. al., PRB **102**, 165136 (2020).

J. Li and M. Eckstein, PRB **103** 045133 (2021).



- ▷ Transient state  $\hat{=}$  NESS
- ▷ NESS  $\leftarrow$  Effective temp + doping level description looks good



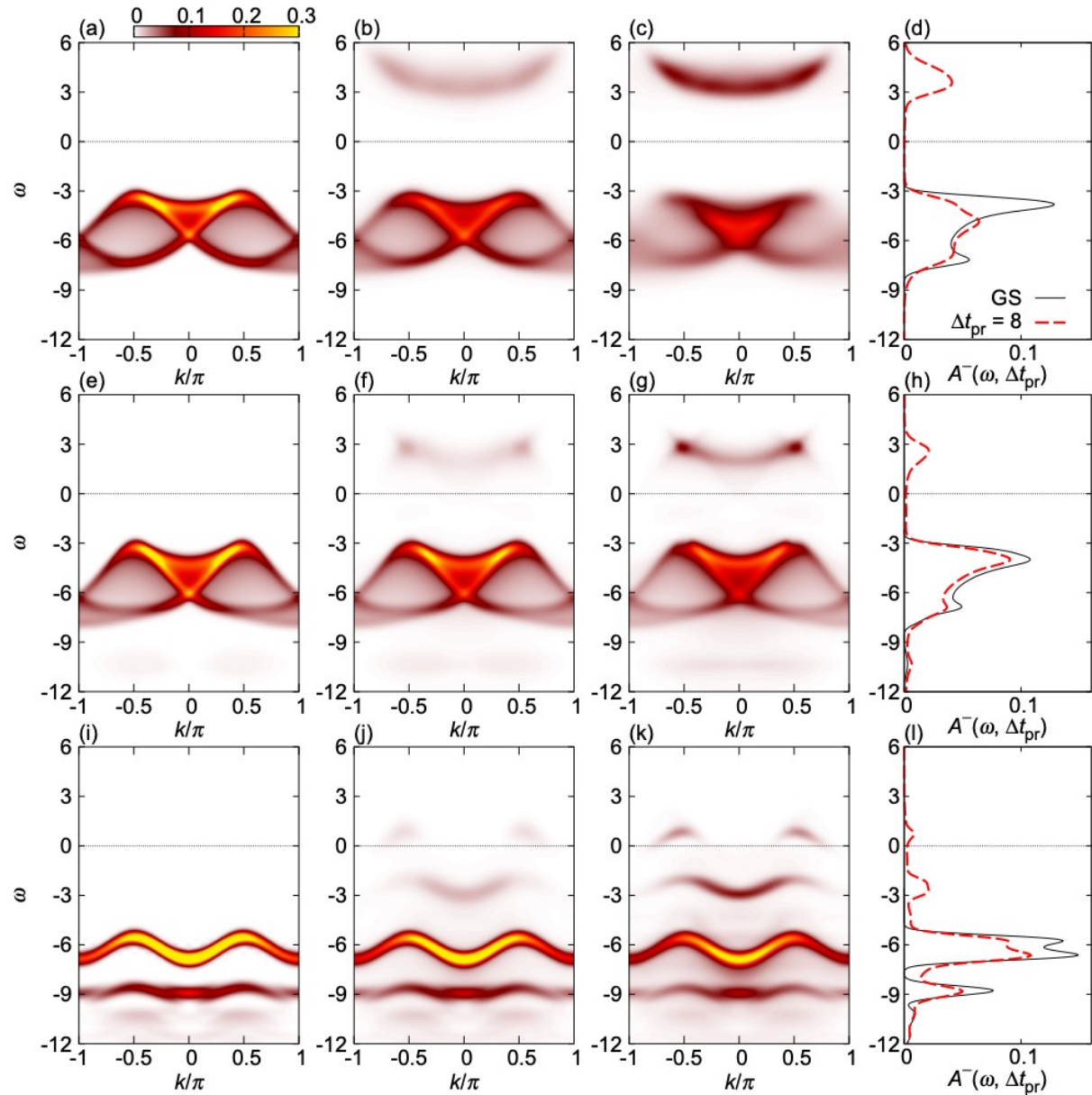


FIG. 5. Calculated single-particle excitation spectra of the 1DEHM at (a), (e), (i)  $\Delta t_{pr} = -\infty$  (GS); (b), (f), (j)  $\Delta t_{pr} = 0$ ; and (c), (g), (k)  $\Delta t_{pr} = 8$ . (d), (h), (l) TDOS at  $\Delta t_{pr} = -\infty$  (black solid line) and  $\Delta t_{pr} = 8$  (red dashed line). The on-site interaction is set to  $U = 10$ , and the intersite interaction, the pump-light frequency, and its intensity are set to (a)-(d)  $V = 0$ ,  $\omega_0 = 8.0$ , and  $A_0 = 0.6$ ; (e)-(h)  $V = 3$ ,  $\omega_0 = 6.04$ , and  $A_0 = 0.3$ ; and (i)-(l)  $V = 6$ ,  $\omega_0 = 6.34$ , and  $A_0 = 0.3$ .

